Respecting priorities versus respecting preferences in school choice: When is there a trade-off?

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Introduction

- Many school districts promote parental choice for schools
- School choice problem consists in assigning students to schools, accounting for students' preferences and school's priorities and capacities
- Choice of algorithm is central to these discussions
- Well-known tradeoff between respecting preferences (Pareto efficiency) and respecting priorities (fairness/envyfreeness).
- Size of the tradeoff is an empirical question.

Introduction

Table 1: Efficiency and envyfreeness across school districts

School district	Algorithms compared	% students with Pareto improving trade	% students with justified envy	Special features
Boston, all levels (Abdulkadiroğlu et al., 2006; Pathak, 2017)	Student-proposing DA, TTC		6.8	Guaranteed placement and sibling priority, catchment area
Budapest, secondary (Biró, 2012; Ortega and Klein, 2022)	Student-proposing DA, TTC		64	Combination of school grades, centralized exam and own school test/interview
Ghent elementary (own source)	School-proposing DA, TTC	< 1	9.2	Sibling and staff priority, distance as tie-breaker
New Orleans - elementary to middle school (Abdulkadiroğlu et al., 2020)	School-proposing DA, TTC		13	Sibling priority, catchment area
New York, high school (Abdulkadiroğlu et al., 2009)	Student-proposing DA, TTC	5.45*	44*	Mix of schools and of priority and ranking criteria

Return * indicates inferred values when these statistics were not directly available.

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- Student-proposing DA and TTC are natural starting points:
  - DA maximizes efficiency among algorithms that produce envy-free outcome (Gale and Shapley, 1962; **?**
  - TTC performs well (and under some circumstances best) among efficient and strategyproof mechanisms (Abdulkadiroğlu et al., 2020; Doğan and Ehlers, 2022; Dur and Paiement, 2022)

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  - TTC performs well (and under some circumstances best) among efficient and strategyproof mechanisms (Abdulkadiroğlu et al., 2020; Doğan and Ehlers, 2022; Dur and Paiement, 2022)
- There is no trade-off between efficiency and envyfreeness when:
  - DA is efficient
  - DA = TTC (more demanding)

What do we know about this research question ?

- **Domain restrictions**
  - **Conditions on priorities** such that DA is efficient (Ergin, 2002; Ehlers and Erdil, 2010; Erdil and Kumano, 2019) or DA and TTC yield the same outcome (Kesten, 2006; Ishida, 2019)
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- **Combination of preferences and priorities**
  - Our condition generalizes Salonen and Salonen (2018), Clark (2006) and Reny (2021)'s conditions

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- **Combination of preferences and priorities**
  - Our condition generalizes Salonen and Salonen (2018), Clark (2006) and Reny (2021)'s conditions
- **Conditions for uniqueness of stable matchings in one-to-one two-sided markets**
  - Alcalde (1994); Eeckhout (2000); Clark (2006); Niederle and Yariv (2009); Legros and Newman (2010); Romero-Medina and Triossi (2013); Lee and Yariv (2014); Gutin et al. (2023)
  - But recall that uniqueness does not imply efficiency

Looking at priorities and preferences together

- **Motivation**: Priorities are not exogenous but expression of what preferences school districts view as legitimate:
  - distance
  - siblings
  - religiosity
  - language programs
- Pathak (2017)'s conjecture

  "Correlation between preferences and priorities induced by proximity may, in turn, result in less scope for Pareto-improving trades across priority groups that involve situations of justified envy. This pattern may then result in a small degree of inefficiency in DA."

Preview of the paper

- We identify a new condition, the **Generalized Mutually Best Pair** (GMBP) condition that captures the degree to which priorities and preferences are congruent
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- Under the GMBP:
    - DA is efficient
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    - DA, school-proposing DA and Immediate Acceptance algo yield the same allocation
- All other situations where DA is efficient are situations with multiple envyfree allocations

Model

- n students with strict preferences $\succ_i$ over schools;
- m schools, with capacity q_s and strict priorities P_s over students;
- A market $\mathcal{E}$ is given by $(\succ, P, q)$;

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- m schools, with capacity q_s and strict priorities P_s over students;
- A market $\mathcal{E}$ is given by $(\succ, P, q)$;
- A matching is a mapping $\mu : I \to S \cup \{s_{m+1}\}$;
- A matching is **feasible** if it does not match more students to a school than its capacity, for all schools;
- A feasible matching μ is (Pareto) **efficient** if there does not exist another feasible matching μ' such that $\mu'(i) \succeq_i \mu(i)$ for all i and strictly so for some i;
- A feasible matching μ is **envyfree** if there does not exist (i, s) such that $s \succ_i \mu(i)$ and $|\mu^{-1}(s)| < q_s \text{ or } iP_sj$ for some $j \in \mu^{-1}(s)$.

Direct priority-based mechanisms

Direct priority-based mechanisms map students' preferences $\succ_i$, school capacities q and priorities P into feasible allocations

- Student-proposing deferred acceptance (DA)
- School-proposing deferred acceptance (school-proposing DA)
- Top-trading cycle (TTC)
- Immediate acceptance (IA) (aka Boston mechanism)

Eeckhout (2000)'s sufficient condition for uniqueness

- **One-to-one** two-sided matching environment
- In a context where all schools have **unit capacity**: there exists a re-ordering of students and schools such that students and schools are the "**mutually best pairs**", i.e. $s_i \succ_i s_k$ for all $k > i$ and $iP_{s_i}k$ for all $k > i$.

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- Eeckhout (2000) shows that this condition ensures uniqueness of the stable matching
- We find that this condition also ensures that DA is efficient and leads to the same outcome as TTC.

Eeckhout's condition is restrictive

- Consider the following market with unit-capacity schools with preferences and priorities as follows:

$$\begin{array}{llll} i_1: & s_1 & s_3 & s_2 \\ i_2: & s_2 & s_1 & \\ i_3: & s_3 & & \end{array} \qquad \begin{array}{lll} s_1: & i_2 & i_1 \\ s_2: & i_1 & i_2 \\ s_3: & i_1 & i_3 \end{array}$$

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i_1 :	s_1	s_3	s_2	s_1 :	i_2	i_1
i_2 :	s_2	s_1		s_2 :	i_1	i_2
i_3 :	s_3			s_3 :	i_1	i_3

- Doesn't satisfy Eeckhout's condition: no mutually best pairs to start with
- Yet, DA is efficient (moreover, DA = TTC and the envyfree matching is unique)

Extending Eeckhout's condition: Simplified market

- Consider the same market

i_1 :	s_1	s_3	~~s_2~~		s_1 :	i_2	i_1
i_2 :	s_2	s_1			s_2 :	~~i_1~~	i_2
i_3 :	s_3				s_3 :	i_1	i_3

- For student 1, school 2 is irrelevant because school 3 is a "safe school" (they're sure to get it if they ask)
- We can remove s_2 from their preferences (and remove s_1 from school 2's priority list)

Simplified market - iterative elimination of irrelevant schools

- Once s_2 has been removed, s_1 becomes irrelevant for student 2, and then s_3 becomes irrelevant for student 1

i_1 :	s_1	s_3	~~s_2~~	s_1 :	~~i_2~~	i_1
i_2 :	s_2	~~s_1~~		s_2 :	~~i_1~~	i_2
i_3 :	s_3			s_3 :	i_1	i_3

- The simplified market satisfies the mutually best pair condition.

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- The simplified market satisfies the mutually best pair condition.
- **Simplified market** $\mathcal{E}^*$ associated with $\mathcal{E}$: given by the outcome of the iterative elimination of irrelevant schools, $\mathcal{E}^* = (\succ^*, P^*, q)$. Definition
  - Preferences are truncated at most preferred "safe school" Priority list P^* is a selection of P

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- **Simplified market** $\mathcal{E}^*$ associated with $\mathcal{E}$: given by the outcome of the iterative elimination of irrelevant schools, $\mathcal{E}^* = (\succ^*, P^*,\ q)$. Definition
  - Preferences are truncated at most preferred "safe school" Priority list P^* is a selection of P
- **Lemma:** The set of envyfree allocations of $\mathcal{E}$ and $\mathcal{E}^*$ are the same

Many-to-one environments

Consider the following market with preferences, capacities and priorities as follows:

$$\begin{array}{llll} i_1: & s_2 & s_1 & \\ i_2: & s_1 & & \\ i_3: & s_1 & s_2 & \\ i_4: & s_1 & s_2 & s_3 \end{array}$$

$$\begin{array}{lllll l} s_1: & i_1 & i_2 & i_3 & i_4 & (\text{capacity} = 2) \\ s_2: & i_2 & i_1 & i_4 & i_3 & (\text{capacity} = 1) \\ s_3: & i_4 & & & & (\text{capacity} = 1) \end{array}$$

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i_2 : s_1
i_3 : s_1 s_2
i_4 : s_1 s_2 s_3

s_1 : i_1 i_2 i_3 i_4 (capacity $= 2$)
s_2 : i_2 i_1 i_4 i_3 (capacity $= 1$)
s_3 : i_4 (capacity $= 1$)

- We cannot further simplify the market.
- Mutually best pair condition not satisfied even if we divide school 1 into two mini-schools

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s_1 :	i_1	i_2	i_3	i_4	(capacity = 2)
s_2 :	i_2	i_1	i_4	i_3	(capacity = 1)
s_3 :	i_4				(capacity = 1)

- We cannot further simplify the market.
- Mutually best pair condition not satisfied even if we divide school 1 into two mini-schools
- Yet, DA is efficient: $(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, s_3)$

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- Yet, DA is efficient: $(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, s_3)$
- Problem: Only top priority student is considered even though schools can admit 2 students!

Many-to-one environments: look at top q_s students

Definition (Sequential Mutually Best Pair condition)

A market $\mathcal{E} = (\succ, P,\ q)$ satisfies the Sequential Mutually Best Pair condition if there is a reordering of students $(i_1, i_2, ...)$ and an associated list of schools S, $(s_{(1)}, s_{(2)}, ...)$, where $s_{(i)} \in S$ stands for the school associated with student i and the same school does not appear more times than its capacity in the list, such that:

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1. $s_{(1)} \succeq_{i_1} s$ for all $s \in S \setminus \{s_{(1)}\}$ and i_1 is among the top $q_{s_{(1)}}$ students in school $s_{(1)}$'s priority list,

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1. $s_{(1)} \succeq_{i_1} s$ for all $s \in S \setminus \{s_{(1)}\}$ and i_1 is among the top $q_{s_{(1)}}$ students in school $s_{(1)}$'s priority list,
2. (for $k > 1$), $s_{(k)} \succeq_{i_k} s$ for all $s \in S^k = \{s \in S : q_s^k > 0\}$, where $q_s^k = q_s - \sum_{l=1}^{k-1} 1_{\{s_{(l)}=s\}}$ is the remaining capacity of school s by the time we reach student i_k, and student i_k is among the top $q_{s_{(k)}}^k$ students in school $s_{(k)}$'s priorities, among students $i_k, i_{k+1}, ...$.

Sequential Mutually Best Pair (SMBP) condition

Revisiting our earlier example:

$$
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$s_1 : \quad i_1 \quad i_2 \quad i_3 \quad i_4 \quad \text{(capacity} = 2)$
$s_2 : \quad i_2 \quad i_1 \quad i_4 \quad i_3 \quad \text{(capacity} = 1)$
$s_3 : \quad i_4 \quad \text{(capacity} = 1)$

- This market satisfies the SMBP condition:

$$\begin{matrix} i_2 & i_2 & i_3 & i_4 \\ s_1 & s_2 & s_1 & s_4 \end{matrix}$$

No trade-off between efficiency and envyfreeness

Definition A market satisfies the **Generalized Mutually Best Pairs** (GMBP) condition if its simplified market $\mathcal{E}^*$ satisfies the SMBP condition.

Proposition 1 Suppose the market $\mathcal{E} = (\succ, P\ , q)$ satisfies the GMBP condition. Then, there is a unique envyfree matching and it is efficient. It is produced by the student-proposing DA, and by (the Nash eqm of) the school-proposing DA and IA.

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Sketch of proof: (1) Establish that the allocation produced by matching sequetially mutually best pairs is the unique envyfree allocation in $\mathcal{E}^*$ (and hence in $\mathcal{E}$), (2) argue that it is efficient, (3) argue that all these mechanisms produce envyfree allocations.

Irrelevance of the algorithm

Proposition 2 If $\mathcal{E} = (\succ, P\ , q)$ satisfies the sequential MBP condition, then TTC, the student-proposing DA, the school-proposing DA and IA (at Nash equilibrium) yield the same allocation and this allocation is both efficient and envyfree.

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Proposition 2 If $\mathcal{E} = (\succ, P \ , q)$ satisfies the sequential MBP condition, then TTC, the student-proposing DA, the school-proposing DA and IA (at Nash equilibrium) yield the same allocation and this allocation is both efficient and envyfree.

Sketch of proof: Show that the allocation produced by the sequential matching of mutually best pairs is also the TTC allocation, using a result from Dur and Paiement (2022) according to which every student weakly prefers their assignment from TTC than any school for which they are among the top q_s students.

GMPB is not necessary for efficiency

Example (all schools have unit capacity)

i_1 :	s_3	s_1			s_1 :	i_1	i_3
i_2 :	s_2				s_2 :	i_2	i_3
i_3 :	s_1	s_2	s_3		s_3 :	i_3	i_1

- This market does not meet GMBP
- Yet, DA is efficient: $(i_1, s_3), (i_2, s_2), (i_3, s_1)$.
- Interestingly, this example is also one where the set of envyfree allocation is not unique.

Non-uniqueness when GMBP is not satisfied and DA is efficient

Proposition 3 School choice market $\mathcal{E} = (\succ,\ P,\ q)$ satisfies the GMBP condition if and only if it admits a unique envyfree allocation and that envyfree allocation is efficient.

Non-uniqueness when GMBP is not satisfied and DA is efficient

Proposition 3 School choice market $\mathcal{E} = (\succ,\ P,\ q)$ satisfies the GMBP condition if and only if it admits a unique envyfree allocation and that envyfree allocation is efficient.

Sketch of proof: By contradiction: suppose that GMBP is not satisfied but there is a unique envyfree allocation that also happens to be efficient.

In the remaining simplified market, students can have a single safe school on their ROLs. Hence the set of priority students at different schools are disjoint and the school-proposing DA will finish after one round.

Argue (by efficiency) that one of the students in this submarket must receive their best choice among schools with remaining capacity. A contradiction.

Relevance in applied contexts

Preferences	Priorities	Application	Ergin's acyclity	MBP everywhere	Sequential MBP	Generalized MBP
Any	Identical priorities (e.g. based on test scores)	University and high school admissions in several countries	✓	✓	✓	✓

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$u_{is} = d_{is} + v_s$	$\pi_{is} = d_{is} + g_i$	Match-quality + common priorities and preferences		✓	✓	✓
Prefers school with sibling, no restriction otherwise	$\pi_{is} = \mathbf{1}_{\{i=\text{sibling}\}} + \varepsilon_i$ ($\varepsilon_i \in [0,1]$)	Sibling priorities, single tie-breaking rule for rest		✓	✓	✓

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Prefers one of their catchment schools, with some exceptions in run-away areas	$\pi_{is} = \mathbf{1}_{\{i=\text{in catchment}\}} + \varepsilon_i$ ($\varepsilon_i \in [0,1]$)	Guaranteed admission in catchment area, single tie-breaking otherwise		✓	✓	✓

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Back to empirical evidence

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Quantifying the tradeoffs

- Assume cardinal utilities underlying student preferences take the following form:

$$u_{is} = \lambda \underbrace{(\delta d_{is} + (1-\delta) v_s)}_{\text{"Structural" part}} + (1-\lambda)\varepsilon_{is},$$

  - d_{is}: student-school match quality (distance, religion, academic inclination);
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  - ε_{is}: idiosyncratic component capturing individual taste

- We consider the following priority structure for schools:

$$\pi_{is} = \alpha \left(\beta d_{is} + (1-\beta) g_i\right) + (1-\alpha)\eta_{is},$$

  - g_i: priorities based on student characteristics and single tie-breaking;
  - η_{is}: residual priorities based on idiosyncratic factors or multiple tie-breaking.

Quantifying the tradeoffs

- A school market environment is characterized by the vector of parameters $(\lambda, \delta, \alpha, \beta)$, the number of schools m, and (equal) school capacities q;
- For each school market, we draw 1,000 realizations of the vector of variables $(d_{is}, v_s, \varepsilon_{is}, g_i, \eta_{is})$ independently from the uniform distribution on $[0, 1]$.

Quantifying the tradeoffs

Table 2: Percentage of markets where DA is efficient, SMBP and GMBP are satisfied

		(1)	(2)	(3)
Preferences	Priorities	DA is efficient	SMBP	GMBP
$\lambda = 1$	$\alpha = 1$	100	100	100
	$\alpha = 0.95$	73.86	64.43	72.25
	$\alpha = 0.90$	15.55	12.24	15.05
$\lambda = 0.75$	$\alpha = 1$	15.94	8.70	14.78
	$\alpha = 0.95$	4.67	0.01	3.42
	$\alpha = 0.90$	0.23	0.00	0.13
$\lambda = 0.5$	$\alpha = 1$	8.08	4.82	6.57
	$\alpha = 0.95$	1.49	0.00	0.41
	$\alpha = 0.90$	0.08	0.00	0.01
$\lambda = 0.25$	$\alpha = 1$	6.23	4.76	5.08
	$\alpha = 0.95$	1.02	0.00	0.10
	$\alpha = 0.90$	0.06	0.00	0.00
$\lambda = 0$	$\alpha = 1$	5.17	4.76	4.79
	$\alpha = 0.95$	0.62	0.00	0.07
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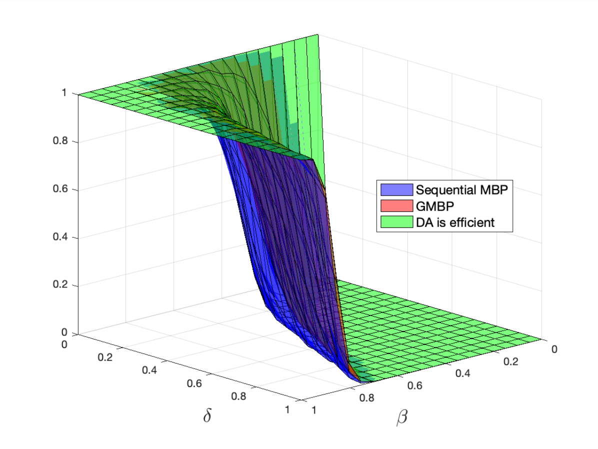
- The ability of DA to generate efficient allocations varies strongly across markets
- Efficiency of DA increases with λ and α (less noisy priorities and preferences)
- GMBP captures a larger fraction of markets where DA is efficient
- Ability of GMBP to capture when DA is efficient increases with λ and α

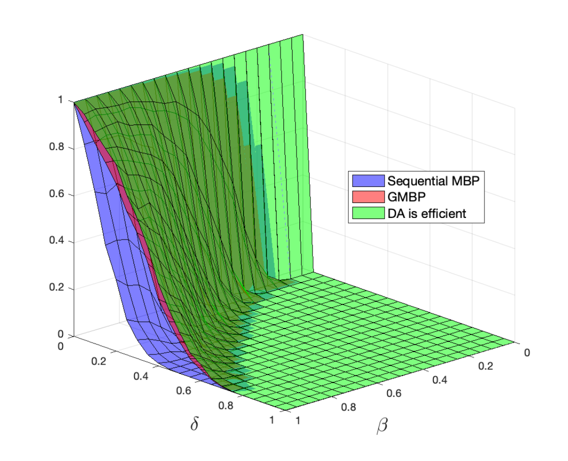
Quantifying the tradeoffs (fraction of markets where DA is efficient)

- A closer look when we fix λ and vary α: $\lambda = 1$, $\alpha = 1$



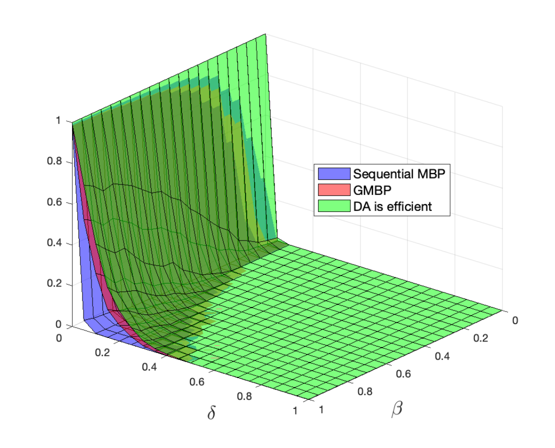
Quantifying the tradeoffs

- A closer look when we fix λ and vary α: $\lambda = 1$, $\alpha = 0.75$



Quantifying the tradeoffs

- A closer look when we fix λ and vary α: $\lambda = 1$, $\alpha = 0.5$



Concluding comments

- When is there a trade-off between preferences (efficiency) and priorities (envyfreeness)?
- **Answer:** If priorities are sufficiently related to preferences as measured by the GMBP condition and in this case the choice of algorithm is second order
  - GMBP maximally captures those markets with a unique and efficient envyfree allocation

Concluding comments

- When is there a trade-off between preferences (efficiency) and priorities (envyfreeness)?
- **Answer:** If priorities are sufficiently related to preferences as measured by the GMBP condition and in this case the choice of algorithm is second order
    - GMBP maximally captures those markets with a unique and efficient envyfree allocation
- **Policy implications:**
    - Understand your market: what drive student preferences and access whether school priorities are likely to be congruent with
    - When designing school choices, increase the probability that your market meets GMBP: use single tie-breaking rule to promote preferences and priority congruence

Empirical evidence

Simplified market

Irrelevant school: Given $\mathcal{E} = (\succ, \ P, \ q)$, school s is irrelevant for student i if there exists a school $s' \neq s$ such that $s' \succ_i s$, and $|j \in I : jP_{s'} i| < q_{s'}$.

Iterative elimination of irrelevant schools.

- Step 1: For each student i, find all their irrelevant schools. If no student has an irrelevant school, stop the process. Otherwise, for each student delete the irrelevant schools from their preferences, and delete the student from the priority list of each irrelevant school.
- Step $k \geq 2$: In the new market with the modified preferences and priorities, repeat Step 1.
- The process finishes when no student has an irrelevant school.

Return

Generalized Mutually Best Pairs

A market $\mathcal{E} = (\succ, P,\ q)$ satisfies the Generalized Mutually Best Pairs (GMBP) condition if there is a reordering of students $(i_1, i_2, ...)$ and an associated list of schools $(s_1, s_2, ...)$ where the same school does not appear more times than its capacity, such that:

1. $s_1 \succ_{i_1} s$ for all $s \in S$ and i_1 is among the top q_{s_1} students in school s_1's priority list,
2. (for $k > 1$), $s_k \succ_{i_k} s$ for all $s \in S_k = \{s \in S : q_s^k > 0\}$, where $q_s^k = q_s - \sum_{l=1}^{k-1} 1_{\{s_l = s\}}$ is the remaining capacity of school s by the time we reach student i_k, and student i_k is among the top $q_{s_k}^k$ students in school s_k, among students $i_k, i_{k+1}, ...$

Return

GBMP not necessary Example

Consider the following market with preferences and priorities as follows (all schools have unit capacity):

$$\begin{array}{llll} i_1: & s_3 & s_1 & \\ i_2: & s_2 & & \\ i_3: & s_1 & s_2 & s_3 \end{array} \qquad \begin{array}{lll} s_1: & i_1 & i_3 \\ s_2: & i_2 & i_3 \\ s_3: & i_3 & i_1 \end{array}$$

- The market does not meet GMBP: while i_2 and s_2 are mutually best, the process stops there.
- Yet, DA is efficient, and yields the same outcome as TTC $(i_1, s_3), (i_2, s_2), (i_3, s_1)$.
- Interestingly, this example is also one where the set of envyfree allocation is not unique.

Return

Mechanisms: Student-proposing deferred acceptance (DA)

Step 1: Each student i proposes to the best school according to $\succ_i$. Each school s provisionally accepts the q_s-highest ranked students, according to P_s, among those students that have proposed to s, and rejects the others.

Step k: Each student i, who has not been previously accepted, proposes to the best school according to $\succ_i$, among those schools that have not yet rejected i. Each school s provisionally accepts the q_s-highest ranked students, according to P_s, among those students that have proposed to s along steps 1 to $k+1$, and rejects the others.

The algorithm terminates at the step where no rejections are made and provisional acceptances become definitive by matching each school s to the set of students provisionally accepted at this step.

Mechanisms: Top-trading cycles (TTC)

Step 1: Each student i points to the best school according to $\succ_i$. If no school is acceptable, i points to s_{m+1} and is removed from the problem. Each school s points to the best student according to P_s, and s_{m+1} points to all students. There exists at least one cycle. Each student i in a cycle is matched to the school s that they point to, in which case i and a seat in s are removed from the problem. If i points to s_{m+1}. They remain unmatched and i is removed from the problem.

Step k**.** Each remaining student i points to the best school according to $\succ_s$, among the schools that still have empty seats. Each school s with an empty seat, points to the best student, according to P_s, among the remaining students, and s_{m+1} points to all of these students. There is at least one cycle. Each student i in a cycle is matched to the school s that they point to, and i and a seat in s are removed from the problem. If i points to s_{m+1}, one remains unmatched and i is removed from the problem.

The algorithm terminates when each student i is either matched to a school or to unassigned.

Mechanisms: Immediate acceptance (IA)

Step 1: Each student i proposes to the best school according to $\succ_i$. Each school s accepts the q_s-highest ranked students, according to P_s, among those students that have proposed to s, and rejects the others. Accepted students are definitive matched to the school. Schools' capacities are reduced by the number of students accepted.

Step k: Each student i, who has not been previously accepted, proposes to the best school according to $\succ_i$, among those available schools that have not yet rejected i. Each school s accepts the q_s-highest ranked students, according to P_s, among those students that have proposed to s, and rejects the others. Accepted students are definitive assigned to the school.Schools' capacities are reduced by the number of students accepted.

The algorithm terminates at the step where no rejections are made.

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