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The effect of bidders' asymmetries on expected revenue in auctions ☆

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Abstract

Bidders' asymmetries are widespread in auction markets. Yet, their impact on behavior and, ultimately, revenue and profits is still not well understood. This paper defines a natural benchmark auction environment to which to compare any private values auction with asymmetrically distributed valuations. The main result is that the expected revenue from the benchmark auction dominates that from the asymmetric auction, both in the first price auction and the second price auction. Moreover, for classes of distributions that lend themselves to a quasi-ordering of more or less asymmetric configurations, we prove that the expected revenue is lower the more asymmetric bidders are. These results formalize the idea that competition is reduced by bidders' asymmetries. Applications to merger analysis, joint bidding and investment are discussed.

JEL classification: D40; D44

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1. Introduction

Known ex ante asymmetries among bidders are widespread in auction markets. For instance, firms with a toehold in the target firm are favored in takeover battles and this advantage is usually understood by all potential buyers. In arts auctions, bidders' tastes are known to be quite idiosyncratic. Asymmetries among bidders have also been documented in procurement markets, with

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sectors ranging from the public works (Bajari, 1998) to the procurement of school milk (Porter and Zona, 1999; Pesendorfer, 2000). In all these auctions, there was one or several firms with a clear comparative advantage over the others.

In this paper, we focus on private values auctions, that is, auctions where at the time when bidders submit their bids, they know how much they value the object they are bidding for.[1] There have been recent advances in our understanding of how these auctions work when bidders' distributions of valuations differ. Most importantly, we now know that an equilibrium exists under quite general conditions in the sealed bid first price auction (Lebrun, 1996; Maskin and Riley, 2000a; Athey, 2001) and understand under what conditions it is unique (Maskin and Riley, 2003; Lebrun, 1999).[2]

Nevertheless, the effect of asymmetries on the auctioneer's expected revenue is still not well understood. Maskin and Riley (2000b) have shown that the revenue ranking between the second price auction and the first price auction depends generally on the kind of asymmetries among bidders. In addition, we know that, in the presence of asymmetries, the first price auction is inefficient (it fails to allocate the object to the highest valuer) and that both the first price auction and the second price auctions are generally suboptimal (they fail to maximize the seller's expected revenue). However, these results shed little light on the impact of bidders' asymmetries within a single market institution: for instance, the first price or the second price auction.

In this paper, we are interested in understanding how (common knowledge) ex ante differences in bidders' distributions of valuations affect their behavior and, in turn, expected revenue and profits. A priori, how asymmetries affect the auctioneer is unclear. In the two auction formats we consider, the auctioneer is the residual claimant of bidders' strategic interactions.[3] The competition among bidders is what determines the winning price, and this is the auctioneer's revenue. When bidders' valuations are asymmetrically distributed, bidders' strategic adjustment to these asymmetries will for sure affect the distribution of social surplus among the bidders and the auctioneer. How so is less clear. In particular, Maskin and Riley (2000b) have shown that "strong" bidders, that is, bidders who are more likely to have a high valuation for the object, prefer a second price auction to a first price auction. This suggests that bidders' attempts to take advantage of their favorable positions might be self-defeating in the first price auction. This could benefit the auctioneer.

To tackle this question, we define a benchmark auction environment to which to compare any auction with asymmetric bidders. An important property of this benchmark is that the distribution of the highest valuation among bidders is the same as in the original auction. In other words, we shall be comparing two auction environments for which the potential social surplus is the same. The question we ask then is: holding the "size of the pie" constant, how do asymmetries affect the share of total surplus that the auctioneer is able to extract?

At a policy level, our analysis is motivated by the following questions. A procurement agency is ready to sponsor cost reducing investments by potential suppliers. The cost reduction is stochastic. For example, suppose as in Tan (1992), that the resulting cost for bidder i is distributed according to the cumulative distribution function $1 - (1 - H(c))^{\alpha_{0i} + \alpha_i}$, where $\alpha_{0i} > 0$ is the initial position, and α_i is the level of investment by bidder i. Suppose furthermore that the unit cost of investment is equal across bidders. Should the procurement agency spend its budget on

[1] More precisely, opponents' signals do not affect a bidder's valuation for the object for sale.

[2] In the second price auction, existence and uniqueness of the dominant strategy equilibrium do not depend on the distributional assumption, so asymmetries do not introduce any difficulty.

[3] In that sense, the auctioneer is in the same position as consumers in a oligopoly market.

a single supplier? Or distribute it across all bidders? As a second example, suppose a firm is considering two different partners for a potential merger. Can we a priori rank these two mergers on the basis of their effect on prices? How does the answer to these questions depend on the resulting market structure?

The main result of the paper is that asymmetries hurt revenue, both in the first price auction (FPA) and in the second price auction (SPA). This result is general in the case of the SPA (Theorems 1 and 2). In the FPA, we obtain analytical results for three classes of distributional asymmetries (Theorems 3–5).

In addition, for the classes of distributions for which a quasi-ordering of potential bidder configurations is available (in the sense: "this configuration of bidder distributions is more asymmetric than this other configuration"), we find that expected revenue is lower the more asymmetric bidders are (Theorems 2 and 3). This includes the kind of asymmetries that arises from mergers, joint bidding or collusion among homogeneous bidders (for example, Graham and Marshall, 1987; Mailath and Zemski, 1991; McAfee and McMillan, 1992, for collusion models; and Tschantz et al., 1997; Waehrer, 1999; Dalkir et al., 2000; or Waehrer and Perry, 2003, for mergers).

Taken together, these results illustrate the decrease in the toughness of competition that market heterogeneity induces, with the following implications for the questions we have raised. First, the procurement agency should distribute its investment subsidies among bidders. It should do so to generate the most symmetric market structure ex post, i.e. set $\alpha_{0i} + \alpha_i = \alpha_{0j} + \alpha_j$, if possible. Second, the merger least harmful for consumer welfare is the one leading to the least asymmetric market structure ex post.

In addition, the results provide some insights on how these specific auction markets work. This is especially useful in the case of the first price auction where the lack of analytical solutions has slowed down our understanding. In particular, the proofs of Theorems 3 and 4 develop analytical techniques that leverage the structure imposed by equilibrium behavior in cases where no analytical solution to the equilibrium is available.

The paper is organized as follows: the next section introduces the model and defines the benchmark environment we will be using. Section 3 studies the equilibrium in the second price auction. Section 4 deals with the first price auction. Section 5 discusses applications as well as an alternative interpretation of the results. Section 6 concludes and suggests venues for future research.

2. A symmetric benchmark

We consider an independent private values auction environment. There are N risk neutral bidders and one object for sale through a sealed bid first price auction or a second price auction. Bidders' valuations are independently distributed with continuously differentiable cumulative distribution functions F_i with support on $[\underline{v}_i, \bar{v}_i]$, $i = 1, \ldots, N$, and positive density everywhere. Valuations are private information but their distributions are common knowledge. Bidders are risk neutral.

Given the selling procedure, the cumulative distributions of bidders' valuations, $(F_1, \ldots, F_N)$, fully characterize the auction environment. We refer to $(F_1, \ldots, F_N)$ as a *configuration*. A configuration is asymmetric, or equivalently, bidders are asymmetric, if $F_i(v) \neq F_j(v)$ for some $i \neq j$ and for a non-zero measure of valuations v.

In this paper, we want to understand how asymmetries affect the outcome in the first price and second price auctions. One way of doing this is to compare the outcome in an asymmetric auction with that of a symmetric auction which, in some sense, we consider a natural point of comparison given our questions and environment.[4]

What properties should this benchmark have? At a conceptual level, an auction is an allocation mechanism. In the private values environment that we consider, the highest level of social surplus (efficiency) is achieved when the object is allocated to the bidder with the highest valuation. A property that seems reasonable to require is that the expected potential social surplus ("the size of the pie") is the same in both the original auction environment and the benchmark environment. This provides a clear interpretation to the result whether revenue in the benchmark environment is lower or greater than that in the asymmetric environment. Indeed, without this condition, we would need to compare the *ratios* of expected revenue to social surplus in order to answer the question of whether asymmetries hurt the auctioneer. Unfortunately, these ratios are not invariant to cardinal changes to the environment.

This requirement is also consistent with the policy experiments we have in mind (distribution of investment efforts, mergers, joint bidding), which change market structure but do not change the potential social surplus in the economy. In fact, these policy experiments are consistent with the stronger requirement that the *distribution* of potential social surplus be the same across configurations.

With these considerations in mind, we introduce the following definition.

Definition 1. Given cumulative distribution functions $(F_1, \ldots, F_N)$, their *symmetric benchmark*, F, is defined, for all v, by

$$F(v) = \left(\prod_{i=1}^{N} F_i(v) \right)^{\frac{1}{N}}. \tag{1}$$

In words, the distribution of valuations in the benchmark environment is the geometric average of the distributions in the original environment. Note that F has support on $[\max_i \underline{v}_i, \max_i \bar{v}_i]$ and is continuously differentiable on its support.

3. The effects of asymmetries in the SPA

In the SPA, the winner is the bidder who places the highest bid and he pays the value of the second highest bid. It is well known that bidding one's own valuation is a dominant strategy in this setting. Hence, the expected revenue in configuration $(F_1, \ldots, F_N)$ is equal to the expected value of the second-order statistics of $(F_1, \ldots, F_N)$. Let S_F denote the cumulative distribution function of the second-order statistics of $(F_1, \ldots, F_N)$. Dropping the arguments, we have

$$S_F = \sum_{i=1}^{N} \left[(1 - F_i) \prod_{j \neq i} F_j \right] + \prod_{i=1}^{N} F_i.$$

[4] We emphasize that any benchmark must depend on the question asked and the environment. For example, Kaplan and Zamir (2002) have recently proposed another benchmark to study asymmetric private values auction. Their benchmark is best suited to answer the following question: suppose the auctioneer has information about the identity of the bidders taking part in the auction (and knowing the identity of a bidder is useful to forecast his/her willingness to pay). Should he reveal this information to the bidders (among others, making them realize the potential asymmetries among them)? Or, should he keep, so to say, all bidders in an equally uninformed state?

Consider a second configuration $(G_1, \ldots, G_N)$, and denote by S_G, the cumulative distribution function of its second-order statistics. Suppose that $\prod_i F_i = \prod_i G_i$. Then

$$S_F - S_G = \sum_{i=1}^{N} \prod_{j \neq i} F_j - \sum_{i=1}^{N} \prod_{j \neq i} G_j. \tag{2}$$

Let $R^s(F_1, \ldots, F_N)$ and $R^s(G_1, \ldots, G_N)$, the expected revenue in the dominant strategy equilibrium of the SPA for configurations $(F_1, \ldots, F_N)$ and $(G_1, \ldots, G_N)$, respectively. The results of this section rely on being able to sign this expression. If we can show that $S_F(v) - S_G(v) \geqslant 0$ for all v, sometimes with strict inequality, then $R^s(F_1, \ldots, F_N) < R^s(G_1, \ldots, G_N)$ follows.

Theorem 1. *Consider any asymmetric configuration of bidders $(F_1, \ldots, F_N)$ and its symmetric benchmark $(F, \ldots, F)$. Then $R^s(F_1, \ldots, F_N) < R^s(F, \ldots, F)$.*

Proof. Letting $(G_1, \ldots, G_N)$ in (2) be equal to $(F, \ldots, F)$, we have

$$S_F - S_G = \sum_i \prod_{j \neq i} F_j - N F^{N-1}. \tag{3}$$

We want to show that this expression is positive for all v. Note that the support of S_G is $[\max_i \underline{v}_i, \max_i \bar{v}_i]$. Thus, for $v \leqslant \max_i \underline{v}_i$, $S_F(v) - S_G(v) \geqslant 0$. For $v > \max_i \underline{v}_i$, $\prod_i F_i(v) > 0$. Dividing the expression in (3) by $N \prod_i F_i$ yields:

$$\frac{1}{N} \sum_i \frac{1}{F_i} - \frac{1}{F}.$$

Because the geometric average is always smaller than the arithmetic average, and strictly so unless all elements are equal, this expression is strictly positive. Thus $S_F(v) - S_G(v) \geqslant 0$ for all v (strictly so for some v when the original configuration is asymmetric) and $R^s(F_1, \ldots, F_N) < R^s(F, \ldots, F)$. □

Theorem 1 has two direct implications:

Corollary 1. *The distribution of revenue in the symmetric benchmark first order stochastically dominates the distribution of revenue in the asymmetric auction.*

Corollary 1 follows directly from the fact that we proved Theorem 1 by showing that $S_F(v) \geqslant S_G(v)$ (sometimes strict).

Corollary 2. *In the second price auction, bidders' ex ante aggregate payoffs from the asymmetric auction $(F_1, \ldots, F_N)$ always dominate that from the symmetric benchmark.*

Corollary 2 follows from the efficiency of the SPA and Theorem 1. Given that the auctioneer is worse off from asymmetries, bidders must benefit.

Theorem 1 concerns the situation in which one configuration is symmetric and the other is asymmetric. In some cases, we can order two asymmetric configurations and argue that one is "more asymmetric" than the other. In those cases, we can generalize Theorem 1 and show that the more asymmetric the configuration, the lower the expected revenue.

One case for which such a partial order is available is for the special but important class of power distributions: $F_i(v) = H(v)^{\alpha_i}$ where H is a continuous cumulative distribution function with support on $[\underline{v}, \bar{v}]$, and $\alpha_i \in \mathbb{R}_+$. These have been used to model efficient collusion, joint bidding and mergers among homogeneous bidders. A configuration in that class of distributions can be represented by a vector of ordered real numbers $(\alpha_1, \ldots, \alpha_N)$ with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_N$. A configuration β is said to be more asymmetric than configuration α if $\sum_{i=1}^m \alpha_i \leqslant \sum_{i=1}^m \beta_i$ (at least one inequality strict) for all $m \leqslant N-1$ and $\sum_{i=1}^N \alpha_i = \sum_{i=1}^N \beta_i$.[5] For example, for two bidders, the β configuration with $\beta_1 > \alpha_1 \geqslant \alpha_2 > \beta_2$ is more asymmetric than configuration α.

Theorem 2. *Consider two configurations of bidders, α and β. In the α configuration, $F_i(v) = H(v)^{\alpha_i}$ for $i = 1, \ldots, N$, and $G_i(v) = H(v)^{\beta_i}$ in the β configuration. If the β configuration is more asymmetric than the α configuration, expected revenue in the dominant strategy equilibrium of the second price auction in the β configuration is strictly lower than in the α configuration.*

Theorem 2 was first proved by Waehrer and Perry (2003). It rationalizes the numerical results presented in Marshall et al. (1994).

Proof. Theorem 2 follows directly from (2). Let $S_\alpha(v)$ denote the cumulative distribution of the second-order statistics in the α configuration, and similarly for $S_\beta(v)$. Dividing expression (2) by $\prod_i F_i$ (which is strictly positive on $(\underline{v}, \bar{v}]$), we get that $S_\beta(v) > S_\alpha(v)$ for $v \in (\underline{v}, \bar{v}]$ if and only if $\sum_i \frac{1}{H^{\beta_i}} > \sum_i \frac{1}{H^{\alpha_i}}$. The proof proceeds by comparing successive configurations of bidders that differ only in the positions of two bidders and by exploiting the fact that $\frac{1}{H^a}$ is a convex function of α. Thus, consider configuration $\gamma = (\beta_1, \alpha_2 + \alpha_1 - \beta_1, \ldots, \alpha_N)$ that differs from configuration α in the positions of bidders 1 and 2. Because $\alpha_1 \leqslant \beta_1$ by assumption and $\beta_1 + \alpha_2 + \alpha_1 - \beta_1 = \alpha_1 + \alpha_2$ by construction, we have $\alpha_1 + \alpha_2 - \beta_1 \leqslant \alpha_2$ and $\sum_i \frac{1}{H^{\gamma_i}} \geqslant \sum_i \frac{1}{H^{\alpha_i}}$ (strictly so if $\alpha_1 < \beta_1$). Next consider configuration $\gamma' = (\beta_1, \beta_2, \alpha_3 + \alpha_1 + \alpha_2 - \beta_2 - \beta_1, \ldots, \alpha_N)$ that differs from configuration γ in the positions of bidders 2 and 3. Because $\alpha_1 + \alpha_2 \leqslant \beta_1 + \beta_2$, we have $\sum_i \frac{1}{H^{\gamma'_i}} \geqslant \sum_i \frac{1}{H^{\gamma_i}}$. We repeat this logic until we reach configuration β. Because configuration β is by assumption more asymmetric, one of the inequalities must be strict. □

More generally, expression (2) suggests that we can rank the revenues from any two configurations that generate the same distribution of potential social surplus whenever $\sum_{i=1}^N \prod_{j \neq i} F_j - \sum_{i=1}^N \prod_{j \neq i} G_j$ has the same sign for all v. When $N = 2$, this condition simplifies to $F_1 + F_2 - (G_1 + G_2)$ having the same sign for all v. This will be satisfied if $\min\{F_1, F_2\} \leqslant G_1, G_2 \leqslant \max\{F_1, F_2\}$ for all v, and $F_1 F_2 = G_1 G_2$. Clearly, configuration (F_1, F_2) is "more asymmetric" than configuration (G_1, G_2).

Theorem 2 can also help us get some intuition for the result. Indeed, while the mechanics behind these results is clear (holding the distribution of the first-order statistics fixed, asymmetries means that the second-order statistics is more likely to be lower), the intuition is not transparent. Consider three bidders, with $\alpha_1 > \alpha_2$ i.e. bidder 1 is more likely to have a high valuation. Suppose bidder 3 is considering merging with either bidder 1 or bidder 2. Merging two bidders

[5] This criterion defines a quasi-ordering. Waehrer and Perry (2003) discuss other measures of asymmetry for this class of distributions. In particular, they show that if configuration β is more asymmetric than configuration α under the criterion above, the Herfindahl index based on expected market share under configuration β is also greater than under configuration α.

means that only the highest of the two valuations will be kept. This will have an effect on revenue only to the extent that the "discarded" valuation was the second highest valuation among all three bidders. We need to consider three cases. First, bidder 3's realization is the highest among the three. Given the distribution, the most likely second highest valuation is bidder 1's so a merger with 2 is preferable for revenue. Second, bidder 3's realization is the second highest. The more likely case that this stays after the merger is that he merges with bidder 2. Finally, bidder 3's valuation is the lowest. In that case, it does not matter with whom he merges since it does not affect the second highest valuation. Thus in all cases, it is better for revenue to have bidder 3 merge with bidder 2. This is also the configuration that involves the lower degree of asymmetries.[6]

4. The effects of asymmetries in the FPA

We now turn to the first price auction. In the FPA, the winner is the bidder who places the highest bid and he pays his own bid. Formally, if bidder i has valuation v_i and wins the auction by submitting a bid b, his resulting payoff is equal to $v_i - b$ (and zero otherwise). Bidders maximize their expected payoff: $\max_{b_i}(v_i - b_i)\Pr(b_i > \max_{j \neq i} b_j)$. A pure strategy equilibrium in this Bayesian game is a vector of bidding functions $b_i : [\underline{v}_i, \bar{v}_i] \to \mathbb{R}_+$, $i = 1, \ldots, N$. A monotone pure strategy equilibrium exists under our assumptions of independent private values and continuously differentiable distribution function on a interval (Lebrun, 1996; Maskin and Riley, 2000a; Athey, 2001; Reny and Zamir, 2004). It is characterized by a set of differential equations with boundary conditions (Plum, 1992; Lebrun, 1999; Maskin and Riley, 2003) unless the supports of the valuations are so disjoint that the equilibrium is degenerate (one bidder always wins). Inverse bid functions $\phi_i : [\underline{b}, \bar{b}_i] \subset \mathbb{R}_+ \to [\underline{v}_i, \bar{v}_i]$, exist and are strictly increasing and continuously differentiable on $(\underline{b}, \bar{b}_i]$. They solve bidders' first-order conditions subject to some boundary conditions for minimum and maximum bids. Finally, Lebrun (1999) and Maskin and Riley (2003) derive conditions under which the equilibrium is unique.

There are three things to point out about the equilibrium in the FPA. First, in general, no analytical solution exists for the equilibrium in the FPA when bidders are asymmetric. Exceptions include the distributions studied in Griesmer et al. (1967) and Plum (1992).[7] Second, the first price auction is generically inefficient when bidders' distributions are asymmetric. Third, when deciding how much to bid, a bidder takes into account the distribution of bids of his opponents only to the extent that his own bid is the highest (i.e. conditional on winning). In other words, what matters for a bidder is the distribution of the second highest bid, conditional on his bid being the highest. By construction, the distribution of the high valuation will be the same across the configurations we consider. So, one interpretation of our results is that this conditional distribution of the second highest bid when bidders are asymmetric induces less aggressive behavior on their part.

We start with an example to illustrate the competitive pressure that a more equal distribution of high realizations among bidders puts on bidding behavior.

Example 1. Suppose there are two bidders with valuations distributed uniformly over $[0, 1]$ (for bidder 1) and $[1, 2]$ (for bidder 2), respectively. Then, bidders never bid more than 1

[6] I thank a referee for suggesting this intuition.

[7] Griesmer et al. consider the case of uniform distributions with a common lower bound, i.e. $F_i(v) = \frac{(v-\underline{v})}{(\bar{v}_i-\underline{v})}$. Plum considers distributions of the form $F_i(v) = \frac{(v-\underline{v})^\alpha}{(\bar{v}_i-\underline{v})^\alpha}$ for $\alpha > 0$.

in equilibrium (by submitting a bid of 1, bidder 2 wins for sure, so he has no incentives to bid higher—bidder 1 does not bid more than her valuation at equilibrium). Therefore, $R^f(F_1, F_2) \leqslant 1$. On the other hand, the benchmark distribution has support on $[1, 2]$. Because this auction satisfies all the conditions of the Revenue Equivalence Theorem (Myerson, 1981; Riley and Samuelson, 1981), we can appeal to this result and conclude that $R^f(F, F)$ is equal to the expected value of the second-order statistics from F, that is, $R^f(F, F) > 1$. Hence, $R^f(F, F) > R^f(F_1, F_2)$.

Example 1 is clearly extreme because the highest valuation is always bidder 2's in the original configuration. Knowing this, bidder 2 is able to shade his bid significantly, and this hurts the auctioneer. By contrast, in the symmetric benchmark (F, F), both bidders are as likely to have the highest valuation and this keeps them on their toes. In the remainder of this section, we show that this result generalizes to less extreme environments.

An important property of some of the bidder configurations we consider is that the distributions can be ranked according to a standard conditional stochastic dominance property (for other uses of this property in auction settings, see, e.g., Tan, 1992; Lebrun, 1999; Waehrer, 1999; Li and Riley, 1999; Maskin and Riley, 2000b; Arozamena and Cantillon, 2004). This imposes the following structure on the equilibrium inverse bid functions and probabilities of winning.

Lemma 1. *(See Lebrun, 1999, Corollary 3; Maskin and Riley, 2000b, Propositions 3.3 and 3.5.*[8]*) Suppose that* $\frac{F_i'}{F_i} > \frac{F_j'}{F_j}$ *for all* $v \in (\max\{\underline{v}_i, \underline{v}_j\}, \min\{\bar{v}_i, \bar{v}_j\}]$ (*in particular, this means that* $F_i < F_j$, *on the interior of their common support—bidder* i *is the most eager bidder*). *Then, at equilibrium*:

(a) $\phi_i(b) > \phi_j(b)$ *for all* b, *on the interior of the intersection of the support of* i *and* j*'s equilibrium bids* (*the "strong" bidder bids less aggressively*); *and*
(b) $F_i(\phi_i(b)) < F_j(\phi_j(b))$ *for all* b, *on the interior of the intersection of the support of* i *and* j*'s equilibrium bids* (*i.e. the "strong" bidder continues to be more likely to win*).

In this section, we prove that asymmetries hurt the auctioneer for three classes of distributions. Moreover, for the class of power distributions, we have a quasi-ordering of configurations and so we are able to prove the stronger result that expected revenue is lower the more asymmetric bidders are. We start with this case.

Theorem 3 *($N = 2$). Suppose that* $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ *for* $\alpha_i, \beta_i \in \mathbb{R}_+$. *Consider two configurations of bidders. In the* α *configuration, bidders' cumulative distributions are* $F_1(v) = H(v)^{\alpha_1}$ *and* $F_2(v) = H(v)^{\alpha_2}$ *where* $H(v)$ *is the cdf of a uniform distribution. In the* β *configuration,* $G_1(v) = H(v)^{\beta_1}$ *and* $G_2(v) = H(v)^{\beta_2}$. *Let* $\alpha_1 \geqslant \alpha_2$. *Then, if* $\beta_1 > \alpha_1$, *the expected revenue from the* β *configuration,* $R^f(\beta)$ *is strictly lower than that from the* α *configuration,* $R^f(\alpha)$.

We provide an outline of the proof here; the full proof is in Appendix A. Denote by (ϕ_1, ϕ_2) the equilibrium inverse bid functions in the α configuration, and by $(\tilde{\phi}_1, \tilde{\phi}_2)$, the equilibrium inverse bid functions in the β configuration. Let $G_\alpha(b) = H(\phi_1(b))^{\alpha_1} H(\phi_2(b))^{\alpha_2}$ and $G_\beta(b)$

[8] Lebrun derives his result under the assumption that the distributions of valuations have a common support $[\underline{v}, \bar{v}]$ with positive density on $(\underline{v}, \bar{v}]$. Maskin and Riley allow different supports but require a positive density on $[\underline{v}_i, \bar{v}_i]$.

$= H(\tilde{\phi}_1(b))^{\beta_1} H(\tilde{\phi}_2(b))^{\beta_2}$, that is, $G_\alpha(b)$ and $G_\beta(b)$ are the cumulative distributions of the winning bids in the α and β configurations, respectively. With these notations, the expressions for the expected revenue in each configuration is given by

$$R^f(\alpha) = \int b\, \mathrm{d}G_\alpha(b), \qquad R^f(\beta) = \int b\, \mathrm{d}G_\beta(b).$$

A sufficient condition for $R^f(\alpha) > R^f(\beta)$ is that $G_\alpha(b) < G_\beta(b)$ on the interior of their common support (first-order stochastic dominance). The minimum equilibrium bid in each configuration, $\underline{b}$, is equal to $\underline{v}$, but we can show that $G_\alpha(b) < G_\beta(b)$ close to $\underline{b}$ (step 1). Next, we use Lemma 1 and bidders' first-order conditions to prove that

$$G_\alpha(b) = G_\beta(b) \quad \Rightarrow \quad \frac{G'_\alpha(b)}{G_\alpha(b)} < \frac{G'_\beta(b)}{G_\beta(b)}.$$

This rules out any crossing of G_α and G_β to the right of $\underline{b}$. We conclude that $G_\alpha(b) < G_\beta(b)$ for all $b > \underline{b}$. Hence, $R^f(\alpha) > R^f(\beta)$.

Theorem 3 is the analog of Theorem 2 for the first price auction when $N = 2$. It rationalizes and generalizes the numerical results reported by Marshall et al. (1994) that greater asymmetries among bidders decreases the auctioneer's expected revenue in the first price auction. As a special case, it also implies that the expected revenue in any asymmetric auction is less than in the symmetric benchmark.

Corollary 3. *Suppose $N = 2$ and $F_i(v) = H(v)^{\beta_i}$ where H is the cdf of a uniform distribution and $\beta_i > 0$. Then $R^f(F_1, F_2) < R^f(F, F)$ as soon as $\beta_1 \neq \beta_2$.*

Proof. Just set $\alpha_1 = \alpha_2 = \frac{\beta_1+\beta_2}{2}$ in Theorem 3. □

As pointed out in Marshall et al. (1994), the class of power distributions has an alternative interpretation in terms of heterogeneous preferences. Suppose that valuations are independently and identically drawn from a uniform distribution on the unit interval and that bidders have a utility function $U_i(x) = x^{k_i}$. Their objective function is given by $(v - b)^{k_i} v$ which is isomorphic to $(v - b)H(v)^{\alpha_i}$ when $H(v) = v$ and $k_i = \frac{1}{\alpha_i}$. Thus Theorem 3 and Corollary 3 implies that in the first price auction, the auctioneer prefers homogeneous risk attitudes over heterogeneous risk attitudes.

Next, we consider asymmetries generated with uniform distributions.

Theorem 4 *($N = 2$). Consider any asymmetric configuration of bidders (F_1, F_2), where F_1 and F_2 are uniform distributions over $[\underline{v}_1, \bar{v}_1]$ and $[\underline{v}_2, \bar{v}_2]$, respectively, where $\underline{v}_1 \leqslant \underline{v}_2$ and $\bar{v}_1 \leqslant \bar{v}_2$. Then $R^f(F_1, F_2) < R^f(F, F)$ as long as the supports of F_1 and F_2 differ.*

The proof can be found in Appendix A. It uses the same logic as the proof of Theorem 3. Note that when $\underline{v}_1 = \underline{v}_2$, Griesmer et al. provide an explicit solution to the equilibrium bidding functions. No known explicit solution is known in the other cases.

Finally, we consider an intermediate class of distributions.

Theorem 5 *($N = 2$). Consider any asymmetric configuration of bidders (F_1, F_2), where $F_i = \frac{(v-\underline{v})^\alpha}{(\bar{v}_i-\underline{v})^\alpha}$, for $\alpha > 0$. Then $R^f(F_1, F_2) < R^f(F, F)$ if $\bar{v}_1 \neq \bar{v}_2$.*

Again, the proof can be found in Appendix A. The logic is similar to the proofs of Theorems 3 and 4. The difference is that, rather than proving that $G^*(b) < G(b)$ close to $\underline{b}$ (the minimum bid is the same in both configurations), we use Plum's (1992) explicit solution for the equilibrium to establish that the maximum equilibrium bid in the benchmark configuration is strictly higher than in the original distribution. Thus $G^*(b) < G(b)$ close to $\bar{b}$.[9]

We end this section by noting that, in all three theorems in this section, our results are actually stronger than simple revenue dominance. In all cases, the distribution of revenue in the symmetric benchmark first-order stochastically dominates the distribution of revenue in the asymmetric auction. This provides an additional empirical prediction.

Corollary 4. *Under the assumptions of Theorems* 3, 4 *or* 5, *the distribution of revenue in the benchmark environment stochastically dominates that of the asymmetric auction.*

5. Discussion

In this section, we revisit the results of the previous sections in light of the two motivations for this research: (1) to compare different market structures in auctions and (2) to better understand, at a conceptual level, the effect of bidders' asymmetries on rents and revenue.

5.1. Applications

Theorems 1–5 provide a partial ordering of market structures in terms of expected revenue. The next examples illustrate how we can use them to evaluate mergers and investment programs in auction markets.

Example 2 *(Bidder sponsoring in a procurement auction with constant returns to scale on investment).* A buyer is facing N suppliers with cumulative cost distribution functions $1-(1-H(x))^{\alpha_{0i}}$ on a common support, $i=1,\dots,N$. The low bidder wins.[10] Each bidder can invest to reduce his cost stochastically according to the process $1-(1-H(x))^{\alpha_{0i}+\alpha_i}$, where α_i is the level of investment. There is a constant marginal cost of investment c.[11] Suppose the buyer has a fixed budget of B to spend on encouraging such investment (for simplicity we assume that the buyer has to spend this budget and covers the entire cost of the investment). Theorem 2 (if the buyer uses a second price auction) and Theorem 3 (if $N=2$ and the buyer uses a first price auction) apply to rank the different sponsoring plans. An investment policy is feasible if $\alpha_i \geqslant 0$, $i=1,\dots,N$, and $c\sum_i \alpha_i = B$. The best feasible investment policy is one such that $\sum_{i=1}^m \alpha_{0i}+\alpha_i \leqslant \sum_{j=1}^m \alpha_{0j}+\hat{\alpha}_j$ for all $m \leqslant N-1$, and for any alternative feasible investment policy $\hat{\alpha}_j$, where bidders are labeled in such a way that $\alpha_{0i}+\alpha_i \geqslant \alpha_{0i+1}+\alpha_{i+1}$ and

[9] This is the only use we can make of Plum's explicit solution. The equilibrium bidding functions cannot be inverted analytically making an explicit computation of expected revenues in both configurations impossible.

[10] This example is framed as a procurement auction, but since procurement auctions and standard auctions are mathematically equivalent, the results apply. The distribution $1-(1-H(x))^{\alpha_{0i}}$ is the procurement analog of distribution $H(x)^{\alpha_{0i}}$. The distribution of the lowest order statistics—which is the relevant measure of social welfare here—is $\prod_i (1-H(x))^{\alpha_{0i}}$.

[11] Returns to scale on investment are said to be constant when the *net* effect of a monetary unit of investment on social surplus is independent of the initial competitive position of the bidder carrying out the investment. Here, returns to scale are constant because: (1) the marginal effect of investment on potential social surplus is the same independently of which bidder carries out the investment, and (2) the marginal cost of investment is constant.

$\alpha_{0j} + \hat{\alpha}_j \geqslant \alpha_{0j+1} + \hat{\alpha}_{j+1}$ (bidders need not be labeled in the same way in the two configurations).[12] Specifically, if there exists a feasible investment policy such that $\alpha_{0i} + \alpha_i = \alpha_{0j} + \alpha_j$ for all i, j, this is the optimal investment policy. Otherwise, Theorems 2 and 3 imply that the investment subsidy should go in priority to the weaker bidders until they "catch up." For concreteness, suppose $\alpha_{01} = 5, \alpha_{02} = 2$ and $\alpha_{03} = 1$ and $B = 3c$, then the optimal investment policy is $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 2$.

Example 2 uses Theorems 2 and 3 literally. The following example combines the insights from these results with the additional partial order provided by first-order stochastic dominance in first and second price auctions. Specifically, if two bidder configurations $(F_1, \ldots, F_N)$ and $(\tilde{F}_1, \ldots, \tilde{F}_N)$ are such that F_i first-order stochastically dominates $\tilde{F}_i$ for all i, then $R^s(F_1, \ldots, F_N) < R^s(\tilde{F}_1, \ldots, \tilde{F}_N)$ (a straightforward consequence of stochastic dominance) and $R^f(F_1, F_2) < R^f(\tilde{F}_1, \tilde{F}_2)$ (Arozamena and Cantillon, 2004, Corollary 1).[13]

Example 3 *(Bidder sponsoring in a procurement auction with decreasing returns to scale on investment).* Consider the same environment as in Example 2 but where marginal costs are an increasing function of bidders' competitive position, $c(\alpha_{0i} + \alpha_i)$ with $c' > 0$. The optimal investment policy is the one that maximizes $\sum_i \alpha_i$ subject to $\sum_i \int_0^{\alpha_i} c(\alpha_{0i} + x)\,\mathrm{d}x = B$. To see this, consider another feasible investment policy β. Thus, $\sum_i \int_0^{\beta_i} c(\alpha_{0i} + x)\,\mathrm{d}x = B$ and $\sum_i \alpha_i > \sum_i \beta_i$ (if $\sum_i \alpha_i = \sum_i \beta_i$, increasing marginal costs and maximal investment mean that α and β are the same policies). Strictly speaking, Theorems 2 and 3 do not apply to compare policy α and policy β. However, consider the policy $\hat{\beta}$ that minimizes asymmetries subject to $\sum_i \hat{\beta}_i = \sum_i \beta_i$. Theorems 2 and 3 imply that $R^s(\hat{\beta}) \geqslant R^s(\beta)$ and $R^f(\hat{\beta}) \geqslant R^f(\beta)$. Now, label bidders such that $\alpha_{0i} + \alpha_i \geqslant \alpha_{0i+1} + \alpha_{i+1}$ and $\alpha_{0i} + \hat{\beta}_i \geqslant \alpha_{0i+1} + \hat{\beta}_{i+1}$ (a bidder's label does not need to be the same in both configurations). By construction, $\alpha_{0i} + \alpha_i \geqslant \alpha_{0i} + \hat{\beta}_i$ for all i (strict for at least one i). Thus, $R^s(\alpha) > R^s(\hat{\beta})$ and $R^f(\alpha) > R^f(\hat{\beta})$. In words, the optimal policy minimizes bidders' asymmetries. The intuition is the same as for Example 2, only reinforced by the fact that it is cheaper to sponsor weaker suppliers. For concreteness, suppose $\alpha_{01} = 5, \alpha_{02} = 2$ and $\alpha_{03} = 1$, $B = 3.75$ and marginal cost $\alpha_{i0} + \alpha_i$ (implying costs equal to $\alpha_{0i}\alpha_i + \frac{1}{2}\alpha_i^2$). The optimal policy is $\alpha_1 = 0$, $\alpha_2 = 0.5$ and $\alpha_3 = 1.5$. It equalizes the marginal costs of investment for bidders 2 and 3 and maximizes the investment level conditional on the budget.

Examples 2 and 3 are reminiscent of Myerson's (1981) optimal auction, which biases the auction allocation in favor of the weak bidders. The mechanism here is different. The auctioneer pays the weak bidders' investment up-front, whether or not they end up winning. In Myerson's auction, the bias can sometimes be implemented as a bidding bonus to the weak bidders—which is relevant only if they win.

Example 4 *(Mergers and joint bidding).* Mergers and joint bidding do not affect the distribution of the first-order statistics, so Theorems 2 and 3 therefore apply directly, with the following

[12] Given that the "more asymmetric than" relation is a quasi-order, one may not be able to rank two arbitrary investment policies. However, the best feasible policy always exists.

[13] Arozamena and Cantillon require $\frac{F_i'}{F_i} < \frac{\tilde{F}_i'}{\tilde{F}_i}$ (strict conditional stochastic dominance), which is satisfied for the distributions studied in Theorems 3 and 4. The distributions in Theorem 5 satisfies weak conditional stochastic dominance.

implications: (1) mergers and joint bidding increase prices in the SPA, and (2) the merger or joint bidding least damaging for prices is the one that involves the two "weakest" bidders.

Example 4 generalizes previous results in the merger literature. They were recently derived for the SPA (Waehrer and Perry, 2003) but have not been established theoretically for the FPA (Marshall et al., 1994; Dalkir et al., 2000, provide numerical results that suggest these two results).

5.2. Asymmetries when the auctioneer can discriminate among bidders

The second motivation for this research was to understand, at a conceptual level, the effect of asymmetries on expected revenue. One possible interpretation of our results is the following: the reason why asymmetries hurt revenue in the SPA and FPA is that the auctioneer is unable to discriminate among bidders in these auction formats. To investigate this conjecture, we consider how asymmetries affect the auctioneer in the (Myerson) optimal auction.

Example 5 *(Asymmetries can benefit the "optimal" auctioneer).* Suppose $N = 2$, F_1 is uniform on $[4, 5]$ and F_2 is uniform on $[4, 8]$. The virtual valuations under (F_1, F_2) are $J_1(v_1) = v_1 - \frac{(1-F_1(v_1))}{f_1(v_1)} = 2v_1 - 5$ and $J_2(v_2) = v_2 - \frac{(1-F_2(v_2))}{f_2(v_2)} = 2v_2 - 8$. Since both virtual valuations are increasing, the problem is regular in the sense of Myerson (1981), and

$$\begin{aligned} R^{\text{opt}}(F_1, F_2) &= \frac{1}{4}\int_4^5\int_4^8 \max\{J_1(v_1), J_2(v_2), 0\}\,\mathrm{d}v_2\,\mathrm{d}v_1 \\ &= \frac{1}{4}\int_4^5\left[\int_4^{v_1+\frac{3}{2}} (2v_1-5)\,\mathrm{d}v_2 + \int_{v_1+\frac{3}{2}}^8 (2v_2-8)\,\mathrm{d}v_2\right]\mathrm{d}v_1 \\ &= \frac{1}{4}\int_4^5\left(\frac{(2v_1-5)^2}{2} + \frac{39}{4} - v_1^2 + 5v_1\right)\mathrm{d}v_1 = 5 + \frac{1}{48}. \end{aligned}$$

The distribution in the benchmark environment is given by

$$F(v) = \begin{cases} \frac{(v-4)}{2}, & v \in [4, 5], \\ \frac{\sqrt{v-4}}{2}, & v \in (5, 8], \end{cases} \quad \text{and} \quad f(v) = \begin{cases} \frac{1}{2}, & v \in [4, 5], \\ \frac{1}{4\sqrt{v-4}}, & v \in (5, 8]. \end{cases}$$

Turning to virtual valuations, we have

$$J(v) = \begin{cases} 2v - 6, & \text{for } v \in [4, 5], \\ 3v - 8 - 4\sqrt{v-4}, & \text{for } v > 5. \end{cases}$$

The virtual valuations are increasing on $[4, 5)$ and on $(5, 8]$. However,

$$\lim_{v\uparrow 5} J(v) = 4 > \lim_{v\downarrow 5} J(v) = 3.$$

This means that the problem is not regular in the sense of Myerson (1981) and the optimal auction requires bunching over an interval of valuations. Consider the following expression:

$$R = \int_4^8 \int_4^8 \max\{J(v_1), J(v_2), 0\} f(v_1) f(v_2)\, \mathrm{d}v_1\, \mathrm{d}v_2. \tag{4}$$

This expression would correspond to the expected revenue from the optimal auction *if* the auction were regular. Given that it is not, (4) *overestimates* the expected revenue from the optimal auction: $R^{\mathrm{opt}}(F, F) < R$. In Appendix B, we explicitly compute (4) and find that $R = 4.8081 < R^{\mathrm{opt}}(F_1, F_2)$. Asymmetries benefit the auctioneer.

6. Conclusions and directions for future research

In this paper, we have sought to understand how ex ante differences in the distributions of bidders' valuations affect revenue and profits. We have shown that, holding the distribution of potential social surplus constant, asymmetries reduce expected revenue, both in the first price and in the second price auctions. In other words, in both cases, asymmetries reduce the share of social surplus that the auctioneer is able to capture. In addition, for the type of bidder heterogeneity that arises from merger, collusion or joint bidding by homogeneous bidders, we have found that the greater the asymmetries, the lower the revenue. Auctions are decentralized allocation mechanisms and the outcome is ultimately driven by bidders' strategic interactions. In that sense, the results formalize the idea that asymmetries reduce the competitive pressure on bidders.

At the policy level, the results can be used for merger analysis in auction markets, the evaluation of the welfare consequences of joint bidding and the evaluation of the impact of market structure on outcomes in general.

Three types of extensions to this research suggest themselves. First, it would be interesting to generalize the analytical results for the FPA to any distribution. Our current approach exploits the structure that equilibrium places on behavior to prove the stronger result that the distribution of revenue in the symmetric benchmark first-order stochastically dominates that in the asymmetric configuration. In numerical experiments using truncated normal distributions, the revenue stochastic dominance condition always held, suggesting both that the result is robust to the choice of distributions and that the way forward is to develop further characterization results for the asymmetric first price auction.

Second, we have made a number of assumptions on the environment, which could be relaxed in the future. Some appear to be simplifying assumptions, for example, our assumption that valuations are independently distributed, or the fact that we have ignored reserve prices. Others are more fundamental and deserve some comments. We have proposed to study asymmetric auctions by comparing the outcome in an asymmetric auction with that of a benchmark symmetric auction. We have argued that the geometric average was the appropriate benchmark in our case. This benchmark is not universal. It depends on both the environment and on the questions to be answered ultimately. For example, the distribution of information rather than the distribution of valuations per se seems to be the relevant dimension for common values environments. As an example of how the choice of a benchmark depends on the question asked, see Kaplan and Zamir (2002). Much remains to be done, but we hope to have demonstrated the use of this approach for understanding market situations with heterogeneity among participants.

Third, an open question is to what extent our results reflect the fact that the two auction mechanisms that we considered are *anonymous* auction mechanisms in the sense that the same rule applies to all bidders. A conjecture is that asymmetries hurt the auctioneer in *any* anonymous auction mechanism. Indeed, as found in Section 5, asymmetries do not necessarily hurt the auctioneer in an optimal auction, i.e. the expected revenue in the asymmetric optimal auction may be higher than in the optimal auction for the benchmark auction. This is left for future research.

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Appendix A

A.1. Proof of Theorem 3

Theorem 3 *($N = 2$). Suppose that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ for $\alpha_i, \beta_i \in \mathbb{R}_+$. Consider two configurations of bidders. In the α configuration, bidders' cumulative distributions are $F_1(v) = H(v)^{\alpha_1}$ and $F_2(v) = H(v)^{\alpha_2}$ where $H(v)$ is the cdf of a uniform distribution. In the β configuration, $G_1(v) = H(v)^{\beta_1}$ and $G_2(v) = H(v)^{\beta_2}$. Let $\alpha_1 \geqslant \alpha_2$. Then, if $\beta_1 > \alpha_1$, the expected revenue from the β configuration, $R^f(\beta)$ is strictly lower than that from the α configuration, $R^f(\alpha)$.*

Proof. Denote by (ϕ_1, ϕ_2) the equilibrium inverse bid functions in the α configuration and by $(\tilde{\phi}_1, \tilde{\phi}_2)$ the equilibrium inverse bid functions in the β configuration (the equilibrium is unique by Lebrun, 1999, Corollary 4). Let $G_\alpha(b)$ be the cumulative distribution function of the winning bids in the α configuration. Define $G_\beta(b)$ similarly. The minimum winning bid is $\underline{b} = \underline{v}$. For future reference, bidders' FOCs in the α configuration read

$$\frac{\alpha_2 H'(\phi_2(b))\phi_2'(b)}{H(\phi_2(b))} = \frac{1}{\phi_1(b) - b}, \tag{A.1}$$

$$\frac{\alpha_1 H'(\phi_1(b))\phi_1'(b)}{H(\phi_1(b))} = \frac{1}{\phi_2(b) - b} \tag{A.2}$$

and similarly for the FOCs in the β configuration.

As a preliminary step, we derive the expression for the derivatives of the inverse bid functions at the origin. Rewriting (A.1), applying l'Hôpital's rule, and appealing to the fact that equilibrium inverse bid functions are continuously differentiable, we have

$$\lim_{b \downarrow \underline{b}} \alpha_2 H'\big(\phi_2(b)\big)\phi_2'(b) = \lim_{b \downarrow \underline{b}} \frac{H(\phi_2(b))}{\phi_1(b) - b} = \frac{H'(\phi_2(\underline{b}))\phi_2'(\underline{b})}{\phi_1'(\underline{b}) - 1}$$

that is,

$$\alpha_2 \phi_2'(\underline{b}) = \frac{\phi_2'(\underline{b})}{\phi_1'(\underline{b}) - 1}. \tag{A.3}$$

Solving for $\phi_1'(\underline{b})$ yields $\phi_1'(\underline{b}) = 1 + \frac{1}{\alpha_2}$. Similarly, $\phi_2'(\underline{b}) = 1 + \frac{1}{\alpha_1}$ and $\tilde{\phi}_i'(\underline{b}) = \frac{1}{\beta_j} + 1$, $j \neq i$.[14]

Step 1. $G_\alpha(b) < G_\beta(b)$ for b close to $\underline{b}$.

At $\underline{b}$, $G_\alpha(\underline{b}) = G_\beta(\underline{b}) = 0$ since $H(\underline{v}) = 0$. We want to show that $\lim_{b \downarrow \underline{b}} \frac{G_\beta(b)}{G_\alpha(b)} > 1$. By the mean value theorem, for all $b > \underline{b}$, there exists $b_1 \in (\underline{b}, b)$ such that:

$$H\big(\phi_1(b)\big) = H'\big(\phi_1(b_1)\big)\phi_1'(b_1)(b - \underline{b})$$

and likewise for $H(\phi_2)$, $H(\tilde{\phi}_1)$ and $H(\tilde{\phi}_2)$ (this defines b_2, $\tilde{b}_1$ and $\tilde{b}_2$). Hence

$$\begin{aligned}
&\lim_{b \downarrow \underline{b}} \frac{G_\beta(b)}{G_\alpha(b)} \\
&\quad \equiv \lim_{b \downarrow \underline{b}} \frac{H(\tilde{\phi}_1(b))^{\beta_1} H(\tilde{\phi}_2(b))^{\beta_2}}{H(\phi_1(b))^{\alpha_1} H(\phi_2(b))^{\alpha_2}} \\
&\quad = \lim_{b \downarrow \underline{b}} \frac{H'(\tilde{\phi}_1(\tilde{b}_1))^{\beta_1} \tilde{\phi}_1'(\tilde{b}_1)^{\beta_1} H'(\tilde{\phi}_2(\tilde{b}_2))^{\beta_2} \tilde{\phi}_2'(\tilde{b}_2)^{\beta_2}}{H'(\phi_1(b_1))^{\alpha_1} \phi_1'(b_1)^{\alpha_1} H'(\phi_2(b_2))^{\alpha_2} \tilde{\phi}_2'(b_2)^{\alpha_2}} \quad \text{(the } (b - \underline{b}) \text{ terms cancel)} \\
&\quad = \frac{[\tilde{\phi}_1'(\underline{b})]^{\beta_1} [\tilde{\phi}_2'(\underline{b})]^{\beta_2}}{[\phi_1'(\underline{b})]^{\alpha_1} [\phi_2'(\underline{b})]^{\alpha_2}} \quad \text{(taking the limit and using } \phi_1(\underline{b}) = \phi_2(\underline{b}) = \tilde{\phi}_1(\underline{b}) = \tilde{\phi}_2(\underline{b})\text{)} \\
&\quad = \frac{(\frac{1}{\beta_1} + 1)^{\beta_2} (\frac{1}{\beta_2} + 1)^{\beta_1}}{(\frac{1}{\alpha_1} + 1)^{\alpha_2} (\frac{1}{\alpha_2} + 1)^{\alpha_1}}.
\end{aligned}$$

We want to show that this expression is greater than 1, or equivalently, that

$$\beta_2 \ln\left(\frac{1}{\beta_1} + 1\right) + \beta_1 \ln\left(\frac{1}{\beta_2} + 1\right) > \alpha_2 \ln\left(\frac{1}{\alpha_1} + 1\right) + \alpha_1 \ln\left(\frac{1}{\alpha_2} + 1\right) \tag{A.4}$$

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[14] This result was first derived by Marshall et al. (1994) who also present plots of the bidding functions. A slight shortcut is involved in this argument because the last step assumes that $\phi_2'(\underline{b})$ is not infinite. A longer and more involved argument is as follows. First note that if $\phi_2'(\underline{b}) = \infty$, then $\phi_1'(\underline{b}) = \infty$ too. Define $\bar{\phi}_2(b) = \phi_2(\underline{b}) + K(b - \underline{b})$ for some $K > 0$ (when $K = \phi_2'(\underline{b})$, $\bar{\phi}_2(b)$ corresponds to the first-order approximation of $\phi_2(b)$ around $\underline{b}$). Define $\bar{b}a_1(v) = \arg\max_b (v - b)H(\bar{\phi}_2(b))^{\alpha_2}$. We have $\frac{\bar{b}_1(v) - \bar{b}_1(\underline{v})}{v - \underline{v}} = \frac{b_1(v) - b_1(\underline{v})}{v - \underline{v}} + \frac{\bar{b}_1(v) - b_1(v)}{v - \underline{v}}$. We will show that $\frac{\bar{b}_1(v) - \bar{b}_1(\underline{v})}{v - \underline{v}} = [1 + \frac{1}{\alpha_2}]^{-1}$ for all v and that $\lim_{v \downarrow \underline{v}} \frac{\bar{b}_1(v) - b_1(v)}{v - \underline{v}} = 0$. Thus, $b_1'(v) \equiv \lim_{v \downarrow \underline{v}} \frac{b_1(v) - b_1(\underline{v})}{v - \underline{v}} = [1 + \frac{1}{\alpha_2}]^{-1}$ as claimed. The argument proceeds in two steps:

1. $\bar{b}_1(v)$ solves the FOC: $\bar{b}_1(v) = v - \frac{H(\bar{\phi}_2(\bar{b}_1(v)))}{\alpha_2 H'(\bar{\phi}_2(\bar{b}_1(v)))K}$. Using the fact that H is the cdf of a uniform distribution, and $\phi_2(\underline{b}) = \underline{v}$, $\frac{H(\bar{\phi}_2(\bar{b}_1(v)))}{H'(\bar{\phi}_2(\bar{b}_1(v)))} = K(\bar{b}_1(v) - \underline{v})$. Thus, $\bar{b}_1(v) = v - \frac{(\bar{b}_1(v) - \underline{v})}{\alpha_2}$ implying $\frac{\bar{b}_1(v) - \bar{b}_1(\underline{v})}{v - \underline{v}} = [1 + \frac{1}{\alpha_2}]^{-1}$ as claimed.
2. Fix any $v > \underline{v}$. From bidders' first-order conditions and taking into account that H is the cdf of a uniform distribution, we have $\bar{b}_1(v) = v - \frac{(\bar{b}_1(v) - \underline{v})}{\alpha_2}$ and $b_1(v) = v - \frac{\phi_2(b_1(v)) - \underline{v}}{\alpha_2 \phi_2'(b_1(v))}$. Thus: $\frac{\bar{b}_1(v) - b_1(v)}{v - \underline{v}} = \frac{\phi_2(b_1(v)) - \underline{v} - \phi_2'(b_1(v))(\bar{b}_1(v) - \underline{v})}{(v - \underline{v})\alpha_2 \phi_2'(b_1(v))}$. Now, $\underline{v} = \phi_2(b_1(\underline{v})) = \phi_2(b_1(v)) + \phi_2'(b_1(v))(b_1(\underline{v}) - b_1(v)) + O^2(b_1(\underline{v}) - b_1(v))$. Substituting this expression leads to $\frac{\bar{b}_1(v) - b_1(v)}{v - \underline{v}} = -[1 + \frac{1}{\alpha_2}]^{-1} \frac{O^2(b_1(\underline{v}) - b_1(v))}{(v - \underline{v})\alpha_2 \phi_2'(b_1(v))} = -[1 + \frac{1}{\alpha_2}]^{-1} \frac{O^2(b_1(\underline{v}) - b_1(v))}{\alpha_2 \phi_1'(b_1(\hat{v}))(b_1(v) - b_1(\underline{v}))\phi_2'(b_1(v))}$ (for some $\hat{v} \in (\underline{v}, v)$ by the mean value theorem). The term $(b_1(v) - b_1(\underline{v}))\phi_2'(b_1(v))$ is in the order of the first-order remainder of a Taylor expansion around $b_1(v)$, $O(b_1(v) - \underline{b})$. Thus the expression converges to zero.

for all $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ and $\beta_1 > \alpha_1 \geqslant \alpha_2 > \beta_2$.

To prove this, we make the following change of variables. Since $\beta_1 > \alpha_1 \geqslant \alpha_2 > \beta_2$, we can express α_1 and α_2 as convex combinations of β_1 and β_2: $\alpha_1 = \lambda_1\beta_1 + (1-\lambda_1)\beta_2$ and $\alpha_2 = \lambda_2\beta_1 + (1-\lambda_2)\beta_2$, where $\lambda_1 + \lambda_2 = 1$ since $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ and $\beta_1 > \beta_2$. Next, define $f(x) = (\alpha_1 + \alpha_2 - x)\ln(\frac{1}{x} + 1)$ with $\alpha_1 + \alpha_2 > x$. The right-hand side of (A.4) can be written as

$$\begin{aligned} f(\alpha_1) + f(\alpha_2) &= f\big(\lambda_1\beta_1 + (1-\lambda_1)\beta_2\big) + f\big(\lambda_2\beta_1 + (1-\lambda_2)\beta_2\big) \\ &< f(\beta_1) + f(\beta_2) \quad \text{given the convexity of } f \text{ and the fact that } \lambda_1 + \lambda_2 = 1. \end{aligned}$$

This corresponds to the left-hand side of (A.4). End of step 1.

Step 2. If $G_\alpha(b) = G_\beta(b)$, for some b in the interior of their support, then $\frac{G'_\alpha(b)}{G_\alpha(b)} < \frac{G'_\beta(b)}{G_\beta(b)}$.

Summing up bidders' FOCs and using the definition of G_α and G_β, we get

$$\frac{G'_\alpha(b)}{G_\alpha(b)} = \frac{1}{\phi_1(b) - b} + \frac{1}{\phi_2(b) - b}, \tag{A.5}$$

$$\frac{G'_\beta(b)}{G_\beta(b)} = \frac{1}{\tilde\phi_1(b) - b} + \frac{1}{\tilde\phi_2(b) - b}. \tag{A.6}$$

We first derive restrictions that equilibrium behavior and the fact that $G_\alpha = G_\beta$ impose on the relationship between $\phi_1, \phi_2, \tilde\phi_1$ and $\tilde\phi_2$. Lemma 1 implies that $\phi_1(b) \geqslant \phi_2(b)$ and $\tilde\phi_1(b) > \tilde\phi_2(b)$ on the interior of their domain. This leaves 6 potential orderings of the equilibrium inverse bid functions at b: (1) $\phi_1 \geqslant \phi_2 \geqslant \tilde\phi_1 > \tilde\phi_2$, (2) $\phi_1 \geqslant \tilde\phi_1 \geqslant \phi_2 \geqslant \tilde\phi_2$ (at least one inequality strict), (3) $\phi_1 \geqslant \tilde\phi_1 > \tilde\phi_2 \geqslant \phi_2$, (4) $\tilde\phi_1 \geqslant \phi_1 \geqslant \tilde\phi_2 \geqslant \phi_2$ (at least one inequality strict), (5) $\tilde\phi_1 \geqslant \phi_1 \geqslant \phi_2 \geqslant \tilde\phi_2$ (at least one inequality strict), and (6) $\tilde\phi_1 > \tilde\phi_2 \geqslant \phi_1 \geqslant \phi_2$.

Now, $G_\alpha(b) = G_\beta(b)$ means that $H(\phi_1(b))^{\alpha_1} H(\phi_2(b))^{\alpha_2} = H(\tilde\phi_1(b))^{\beta_1} H(\tilde\phi_2(b))^{\beta_2}$. Let $\lambda = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\tilde\lambda = \frac{\beta_1}{\beta_1+\beta_2}$. This expression can be rewritten as

$$\begin{aligned} &\lambda \ln H\big(\phi_1(b)\big) + (1-\lambda)\ln H\big(\phi_2(b)\big) \\ &\quad = \tilde\lambda \ln H\big(\tilde\phi_1(b)\big) + (1-\tilde\lambda)\ln H\big(\tilde\phi_2(b)\big) \quad \text{with } \lambda < \tilde\lambda. \end{aligned} \tag{A.7}$$

This rules out orderings (1), (4) and (6). The following claim rules out ordering (3).

Claim 1. *There cannot be any value of bid $\hat b > \underline{b}$ for which $\phi_1(\hat b) \geqslant \tilde\phi_1(\hat b) > \tilde\phi_2(\hat b) \geqslant \phi_2(\hat b)$.*

Proof. We first claim that if there exists such a $\hat b$, then

$$\phi_1(b) > \tilde\phi_1(b) \geqslant \tilde\phi_2(b) > \phi_2(b) \quad \text{for all } b > \hat b. \tag{A.8}$$

The only way in which (A.8) could be false is if there exists a b such that (1) $\phi_1(b) = \tilde\phi_1(b)$ and $\phi'_1(b) \leqslant \tilde\phi_1(b)$ while $\tilde\phi_2(b) \geqslant \phi_2(b)$, or (2) $\phi_2(b) = \tilde\phi_2(b)$ and $\phi'_2(b) \geqslant \tilde\phi_2(b)$ while $\tilde\phi_1(b) \leqslant \phi_1(b)$. In the first case, we have

$$\alpha_1 \frac{H'(\phi_1)}{H(\phi_1)}\phi'_1 < \beta_1 \frac{H'(\tilde\phi_1)}{H(\tilde\phi_1)}\tilde\phi'_1 \quad \text{since } \alpha_1 < \beta_1$$

which, using bidder 2's FOC in both configurations (Eq. (A.2) and its equivalent) implies that $\phi_2 > \tilde\phi_2$, a contradiction. In the second case, we have

$$\alpha_2 \frac{H'(\phi_2)}{H(\phi_2)}\phi'_2 > \beta_2 \frac{H'(\tilde\phi_2)}{H(\tilde\phi_2)}\tilde\phi'_2 \quad \text{since } \alpha_2 > \beta_2$$

which, using bidder 1's FOC in both configurations implies that $\phi_1 < \tilde{\phi}_1$, a contradiction. Thus, (A.8) must hold.

We are now ready to reach a contradiction. With two bidders, the equilibrium maximum bid is common to both bidders (e.g., Lebrun, 1999). Let us denote them by $\bar{b}_\alpha$ (for the maximum bid in the α configuration) and $\bar{b}_\beta$, respectively. Since the upper bound to the distributions of valuations, $\bar{v}$, is common in both configurations, we must also have $\phi_1(\bar{b}_\alpha) = \tilde{\phi}_1(\bar{b}_\beta) = \tilde{\phi}_2(\bar{b}_\beta) = \phi_2(\bar{b}_\alpha) = \bar{v}$. This is impossible if (A.8) holds. This proves the claim. □

We conclude that only orderings (2) and (5) are possible when $G_\alpha = G_\beta$. We now show that $\frac{G'_\alpha}{G_\alpha} < \frac{G'_\beta}{G_\beta}$ must hold. Referring back to (A.5) and (A.6), $\frac{G'_\alpha}{G_\alpha} < \frac{G'_\beta}{G_\beta}$ follows trivially with ordering (2). The next claim proves that $\frac{G'_\alpha}{G_\alpha} < \frac{G'_\beta}{G_\beta}$ if ordering (5) holds.

Claim 2. *Suppose there exists a b on the interior of the bidders' bid supports such that $G_\alpha(b) = G_\beta(b)$ and $\tilde{\phi}_1(b) \geqslant \phi_1(b) \geqslant \phi_2(b) \geqslant \tilde{\phi}_2(b)$ (at least one inequality strict). Then $\frac{G'_\alpha}{G_\alpha} < \frac{G'_\beta}{G_\beta}$.*

Proof. Let $f(x) = \ln H(x)$, $\lambda = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\tilde{\lambda} = \frac{\beta_1}{\beta_1+\beta_2}$, and consider the following optimization problem:

$$\max_{\phi_1,\phi_2,\tilde{\phi}_1,\tilde{\phi}_2} \frac{1}{\phi_1 - b} + \frac{1}{\phi_2 - b} - \frac{1}{\tilde{\phi}_1 - b} - \frac{1}{\tilde{\phi}_2 - b} \tag{A.9}$$

subject to

$$\lambda f(\phi_1) + (1-\lambda) f(\phi_2) = \tilde{\lambda} f(\tilde{\phi}_1) + (1-\tilde{\lambda}) f(\tilde{\phi}_2) \quad (\gamma), \tag{A.10}$$

$$\tilde{\phi}_1 \geqslant \phi_1 \quad (\delta_1), \tag{A.11}$$

$$\phi_1 \geqslant \phi_2 \quad (\delta_2). \tag{A.12}$$

The first constraint corresponds to the requirement that $G_\alpha = G_\beta$. The second and third constraints are the *relaxed* version of the requirement on the ordering of the inverse bid functions (note that (A.10) to (A.12) imply $\phi_2 \geqslant \tilde{\phi}_2$—cf. the argument around (A.7)). The corresponding multipliers are in parenthesis.

The objective function corresponds to $\frac{G'_\alpha}{G_\alpha} - \frac{G'_\beta}{G_\beta}$. The idea of the proof is to show that the value of this objective function at the optimum is negative.

This is an optimization problem of a continuous function over a compact set, so a solution exists. The first-order conditions are given by

$$-\frac{1}{(\phi_1 - b)^2} + \gamma\lambda f'(\phi_1) - \delta_1 + \delta_2 = 0,$$
$$-\frac{1}{(\phi_2 - b)^2} + \gamma(1-\lambda) f'(\phi_2) - \delta_2 = 0,$$
$$\frac{1}{(\tilde{\phi}_1 - b)^2} - \gamma\tilde{\lambda} f'(\tilde{\phi}_1) + \delta_1 = 0,$$
$$\frac{1}{(\tilde{\phi}_2 - b)^2} - \gamma(1-\tilde{\lambda}) f'(\tilde{\phi}_2) = 0.$$

Solving for γ, we get

$$\frac{1+(\delta_1-\delta_2)(\phi_1-b)^2}{\lambda f'(\phi_1)(\phi_1-b)^2}=\frac{1+\delta_2(\phi_2-b)^2}{(1-\lambda)f'(\phi_2)(\phi_2-b)^2}=\frac{1+\delta_1(\tilde{\phi}_1-b)^2}{\tilde{\lambda}f'(\tilde{\phi}_1)(\tilde{\phi}_1-b)^2}$$
$$=\frac{1}{(1-\tilde{\lambda})f'(\tilde{\phi}_2)(\tilde{\phi}_2-b)^2}. \tag{A.13}$$

Since $f(v)=\ln H(v)$, $(v-b)^2 f'(v)=\frac{(v-b)^2}{(v-\underline{v})}$ is strictly increasing in v. Using the fact that $\tilde{\lambda}>\lambda\geqslant 1-\lambda>1-\tilde{\lambda}$, we can argue that any candidate solution to (A.13) must be such that $\delta_1>0$ and $\delta_2>0$, that is (because $G_\alpha=G_\beta$), $\phi_1=\phi_2=\tilde{\phi}_1=\tilde{\phi}_2$. Indeed, consider first the third and fourth expressions in (A.13). Since $\tilde{\lambda}f'(\tilde{\phi}_1)(\tilde{\phi}_1-b)^2>(1-\tilde{\lambda})f'(\tilde{\phi}_2)(\tilde{\phi}_2-b)^2$, $\delta_1>0$. Next, consider the second and fourth expressions. Since $(1-\lambda)f'(\phi_2)(\phi_2-b)^2>(1-\tilde{\lambda})f'(\tilde{\phi}_2)(\tilde{\phi}_2-b)^2$, $\delta_2>0$ must hold.

Thus, the KT conditions imply that $\phi_1=\phi_2=\tilde{\phi}_1=\tilde{\phi}_2$ is the only candidate solution to the problem of maximizing (A.9) subject to (A.10) to (A.12). The Kuhn–Tucker conditions are necessary for an optimum when the constraint qualification is satisfied. Here, the constraint qualification comes down to the linear independence of the vectors in

$$\begin{bmatrix}\lambda & (1-\lambda) & -\tilde{\lambda} & -(1-\tilde{\lambda})\\ -1 & 0 & 1 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}.$$

The matrix is of rank 3 so the constraint qualification is satisfied.

Finally, we need to show that any such solution corresponds to a maximum. To do so, consider a small deviation from $\phi_1=\phi_2=\tilde{\phi}_1=\tilde{\phi}_2$, that satisfies $G_\alpha=G_\beta$ and

$$\tilde{\phi}_1(b)\geqslant\phi_1(b)\geqslant\phi_2(b)\geqslant\tilde{\phi}_2(b). \tag{A.14}$$

Differentiating totally $\lambda f(\phi_1)+(1-\lambda)f(\phi_2)-\tilde{\lambda}f(\tilde{\phi}_1)-(1-\tilde{\lambda})f(\tilde{\phi}_2)=0$ implies $d\tilde{\phi}_2=\frac{1}{1-\tilde{\lambda}}[\lambda d\phi_1+(1-\lambda)d\phi_2-\tilde{\lambda}d\tilde{\phi}_1]$. Therefore,

$$d\left[\frac{1}{\phi_1-b}+\frac{1}{\phi_2-b}-\frac{1}{\tilde{\phi}_1-b}-\frac{1}{\tilde{\phi}_2-b}\right]$$
$$=\frac{1}{(\phi_1-b)^2}\left[-d\phi_1-d\phi_2+d\tilde{\phi}_1+d\tilde{\phi}_2\right]$$
$$=\frac{1}{(\phi_1-b)^2}\left[\frac{\lambda+\tilde{\lambda}-1}{1-\tilde{\lambda}}d\phi_1+\frac{\tilde{\lambda}-\lambda}{1-\tilde{\lambda}}d\phi_2-\frac{(2\tilde{\lambda}-1)}{1-\tilde{\lambda}}d\tilde{\phi}_1\right].$$

From (A.14) and the fact that $\tilde{\lambda}>\lambda\geqslant\frac{1}{2}$,

$$\frac{\lambda+\tilde{\lambda}-1}{1-\tilde{\lambda}}d\phi_1+\frac{\tilde{\lambda}-\lambda}{1-\tilde{\lambda}}d\phi_2-\frac{(2\tilde{\lambda}-1)}{1-\tilde{\lambda}}d\tilde{\phi}_1\leqslant\frac{\lambda-\tilde{\lambda}}{1-\tilde{\lambda}}d\phi_1+\frac{\tilde{\lambda}-\lambda}{1-\tilde{\lambda}}d\phi_2\leqslant 0.$$

Thus, the net effect of any perturbation around the candidate solution is negative (strictly so, unless the perturbation is such that $d\phi_1=d\phi_2=d\tilde{\phi}_1=d\tilde{\phi}_2$). We conclude $\phi_1=\phi_2=\tilde{\phi}_1=\tilde{\phi}_2$ is a local (and global, given that there is no other candidate solution) optimum. The value of the objective function at the optimum is equal to zero. Since the real constraint on the inverse bid functions is actually stronger that the inequalities $\tilde{\phi}_1\geqslant\phi_1\geqslant\phi_2\geqslant\tilde{\phi}_2$ (one inequality at least must be strict), we conclude that $\frac{1}{\phi_1-b}+\frac{1}{\phi_2-b}-\frac{1}{\tilde{\phi}_1-b}-\frac{1}{\tilde{\phi}_2-b}$ is always strictly negative when $G_\alpha=G_\beta$, that is $\frac{G'_\alpha}{G_\alpha}<\frac{G'_\beta}{G_\beta}$. End of proof of claim 2. This also concludes step 2.

Steps 1 and 2 together imply that $G_\alpha(b) < G_\beta(b)$ for all b on the interior of their common support. Therefore, $R^f(\beta) < R^f(\alpha)$. □

A.2. Proof of Theorem 4

Theorem 4 *($N = 2$). Consider any asymmetric configuration of bidders (F_1, F_2) where F_1 and F_2 are uniform distributions over $[\underline{v}_1, \bar{v}_1]$ and $[\underline{v}_2, \bar{v}_2]$ respectively, where $\underline{v}_1 \leqslant \underline{v}_2$ and $\bar{v}_1 \leqslant \bar{v}_2$. Then $R^f(F_1, F_2) < R^f(F, F)$ as long as the supports of F_1 and F_2 differ.*

Proof. We need to consider two cases.

Case 1. $\bar{v}_1 < \underline{v}_2$ and the equilibrium is degenerate, i.e. in any equilibrium, bidder 2 bids $\bar{v}_1$ for all realizations of his valuation. Bidder 1 never wins and submit bids on $[\underline{v}_1, \bar{v}_1]$ that make bidding $\bar{v}_1$ a best response for bidder 2 (Example 1 was a special case of such a degenerate equilibrium). In that case, $R^f(F_1, F_2) = \bar{v}_1 < R^f(F, F)$ since F has support over $[\underline{v}_2, \bar{v}_2]$.

Case 2. The equilibrium is non-degenerate. In that case, it is characterized by a system of differential equations with boundary conditions. The rest of the proof deals with this case.

Let (ϕ_1, ϕ_2) denote the equilibrium inverse bid function under (F_1, F_2) (with support on $[\underline{b}, \bar{b}]$) and let (ϕ, ϕ) denote the equilibrium inverse bid functions in the benchmark auction (with support on $[\underline{b}^*, \bar{b}^*]$). Let $G(b) = F_1(\phi_1(b))F_2(\phi_2(b))$ and $G^*(b) = F(\phi(b))^2$.

Step 1. $\underline{b} \leqslant \underline{b}^*$ and $G(b) > G^*(b)$ in a neighborhood to the right of $\underline{b}^*$.

Proof. When $\underline{v}_1 < \underline{v}_2$, $\underline{b} \in (\underline{v}_1, \underline{v}_2)$ (Maskin and Riley, 2003, Lemma 3) and $\underline{b}^* = \underline{v}_2$ (the minimum valuation of the benchmark distribution). Hence, $G(b) > G^*(b)$ to the right of $\underline{b}^*$. When $\underline{v}_1 = \underline{v}_2$, the claim follows from the analytical solution presented in Griesmer et al. (1967). Without loss of generality, let $\underline{v}_1 = \underline{v}_2 = 0$. Griesmer et al. show that, for $\bar{v}_2 > \bar{v}_1$, the inverse bidding functions are given by $\phi_1(b) = \frac{2b}{1+Cb^2}$ and $\phi_2(b) = \frac{2b}{1-Cb^2}$ where $C = \frac{\bar{v}_2^2 - \bar{v}_1^2}{\bar{v}_1^2 \bar{v}_2^2} > 0$.[15] Thus $G(b) = \frac{1}{\bar{v}_1\bar{v}_2}\frac{4b^2}{1-C^2b^4}$. On $[0, \bar{v}_1]$, the benchmark distribution is given by $F(v) = \frac{v}{\sqrt{\bar{v}_1\bar{v}_2}}$. Using the well-known solution for the symmetric FPA, $b(v) = \frac{1}{F(v)}\int_0^v x\,\mathrm{d}F(x) = \frac{v}{2}$ on $[0, \bar{v}_1]$. Thus, $G^*(b) = \frac{1}{\bar{v}_1\bar{v}_2}4b^2$. The claim follows because $\frac{1}{\bar{v}_1\bar{v}_2}\frac{4b^2}{1-C^2b^4} > G^*(b) = \frac{1}{\bar{v}_1\bar{v}_2}4b^2$. End of step 1. □

Step 2. $G(b) = G^*(b)$ for $b > \underline{b} \Rightarrow \frac{G'(b)}{G(b)} > \frac{G^{*'}(b)}{G^*(b)}$ as long as $\phi_1(b) < \phi_2(b)$.

Proof. By assumption, $F_1(v) = \frac{v-\underline{v}_1}{\bar{v}_1-\underline{v}_1}$, $F_2(v) = \frac{v-\underline{v}_2}{\bar{v}_2-\underline{v}_2}$ and $F(v)^2 \leqslant \frac{(v-\underline{v}_1)(v-\underline{v}_2)}{(\bar{v}_1-\underline{v}_1)(\bar{v}_2-\underline{v}_2)}$.[16] Hence, $G(b) = G^*(b)$ implies

$$\big(\phi_1(b) - \underline{v}_1\big)\big(\phi_2(b) - \underline{v}_2\big) \leqslant \big(\phi(b) - \underline{v}_1\big)\big(\phi(b) - \underline{v}_2\big). \tag{A.15}$$

Using bidders' first-order conditions, the inequality $\frac{G'(b)}{G(b)} > \frac{G^{*'}(b)}{G^*(b)}$ can be rewritten as

$$\frac{1}{\phi_1(b) - b} + \frac{1}{\phi_2(b) - b} > \frac{2}{\phi(b) - b}. \tag{A.16}$$

[15] Griesmer et al.'s solution is reproduced in a separate appendix available at http://www.ssrn.com and at http://www.ecares.org/ecantillon.html.

[16] $F(v)^2 = \frac{(v-\underline{v}_1)(v-\underline{v}_2)}{(\bar{v}_1-\underline{v}_1)(\bar{v}_2-\underline{v}_2)}$ when $v \leqslant \bar{v}_1$.

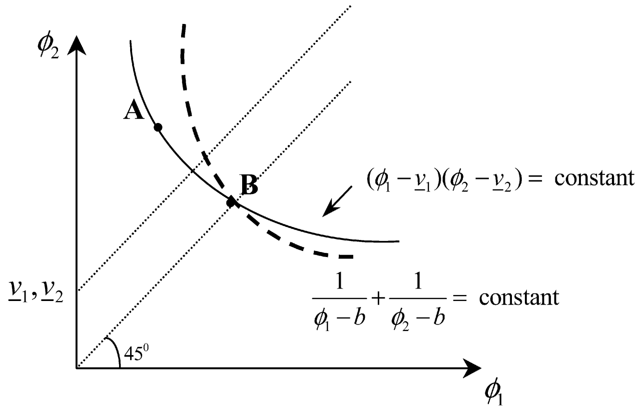

Fig. 1. Iso-level curves for f and g.

We want to show that (A.15) implies (A.16) as long as $\phi_1(b) < \phi_2(b)$. First note that $f(\phi_1, \phi_2) = (\phi_1 - \underline{v}_1)(\phi_2 - \underline{v}_2)$ has a convex upper contour set and that $g(\phi_1, \phi_2) = \frac{1}{\phi_1 - b} + \frac{1}{\phi_2 - b}$ has a convex lower contour set. Moreover, f is symmetric around the 45° line through $(\underline{v}_1, \underline{v}_2)$ and $g(\phi_1, \phi_2)$ is symmetric around the origin. In particular, this means that the slope of the iso-level curve $f(\phi_1, \phi_2)$ at the 45° line through $(\underline{v}_1, \underline{v}_2)$ is equal to -1, and the slope of the iso-level curve of $g(\phi_1, \phi_2)$ through the 45° line from the origin is equal to -1. Finally, whenever the iso-level curves f and g cross at $\phi_1 < \phi_2$, the slope of the iso-level curve g is steeper than the slope of the iso-level curve f. Indeed, consider the following set of inequalities:

$$\left|\frac{d\phi_2}{d\phi_1}\right|_f = \frac{\phi_2 - \underline{v}_2}{\phi_1 - \underline{v}_1} \leqslant \frac{\phi_2 - \underline{b}}{\phi_1 - \underline{b}} < \frac{\phi_2 - b}{\phi_1 - b} < \left(\frac{\phi_2 - b}{\phi_1 - b}\right)^2 = \left|\frac{d\phi_2}{d\phi_1}\right|_g \tag{A.17}$$

where the first inequality comes from $\underline{v}_1 \leqslant \underline{b} \leqslant \underline{v}_2$, the second inequality comes from the fact that $\frac{\phi_2 - b}{\phi_1 - b}$ is increasing in b when $\phi_1 < \phi_2$, and the last inequality follows from $\phi_2 > \phi_1$. These are represented in Fig. 1.

Consider any value for the pair (ϕ_1, ϕ_2) such that $\phi_1 < \phi_2$, for example, point A in Fig. 1. Then the lowest value of ϕ consistent with (A.15) is at point B, and $(A.17)$ ensures that B lies on a lower iso-level curve of g than A.[17] Hence (A.16) must hold. End of step 2. □

From Lemma 1, we know that $\phi_1(b) < \phi_2(b)$ on $(\underline{b}, \bar{b})$. Thus step 1 and step 2 together imply that $G^*(b) < G(b)$ for all $b \in (\underline{b}, \bar{b})$. $R^f(F_1, F_2) < R^f(F, F)$ follows. □

A.3. Proof of Theorem 5

Theorem 5 *($N = 2$). Consider any asymmetric configuration of bidders (F_1, F_2), where $F_i = \frac{(v - \underline{v})^\alpha}{(\bar{v}_i - \underline{v})^\alpha}$, for $\alpha > 0$. Then $R^f(F_1, F_2) < R^f(F, F)$ if $\bar{v}_1 \neq \bar{v}_2$.*

[17] When $\underline{v}_1 = \underline{v}_2$, the upper contour set of g is included in the lower contour set of f.

Proof. Without loss of generality, let $\underline{v}=0$, $\bar{v}_2=1$ and $\bar{v}_1\in(0,1)$. Plum (1992) has shown that the equilibrium bidding functions are given by

$$b_i(v)=\frac{1-(1-c_i v^{\alpha+1})^{\frac{\alpha}{\alpha+1}}}{c_i v^{\alpha}}\quad \text{where } c_i=\frac{1}{\bar{v}_i{}^{\alpha+1}}-\frac{1}{\bar{v}_j{}^{\alpha+1}},\ i\neq j. \tag{A.18}$$

The benchmark distribution is given by

$$F(v)=\begin{cases}\frac{v^{\alpha}}{\bar{v}_1{}^{\frac{\alpha}{2}}} & \text{for } v\in[0,\bar{v}_1],\\ v^{\frac{\alpha}{2}} & \text{for } v\in(\bar{v}_1,1].\end{cases}$$

Let $\bar{b}(\bar{v}_1,\alpha)$ and $\bar{b}^*(\bar{v}_1,\alpha)$ denote the upper bound to equilibrium bids in configuration (F_1,F_2) and (F,F), respectively. The lower bound to equilibrium bids is the same in both configurations and denoted $\underline{b}=0$.

Claim 1. *$\bar{b}(\bar{v}_1,\alpha)<\bar{b}^*(\bar{v}_1,\alpha)$ for all $\alpha>0$ and $\bar{v}_1\in(0,1)$.*

Proof of Claim 1. Using the explicit solution for an equilibrium in a symmetric auction,

$$\bar{b}^*(\bar{v}_1,\alpha)=\int_0^1 x\,\mathrm{d}F(x)=1-\int_0^{\bar{v}_1}F(x)\,\mathrm{d}x-\int_{\bar{v}_1}^1 F(x)\,\mathrm{d}x=\frac{\alpha}{\alpha+2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\bar{v}_1^{\frac{\alpha}{2}+1}.$$

From (A.18), $\bar{b}(\bar{v}_1,\alpha)=\frac{\bar{v}_1(1-\bar{v}_1^{\alpha})}{(1-\bar{v}_1^{\alpha+1})}$. It is easy to check that $\lim_{\bar{v}_1\uparrow 1}\bar{b}^*(\bar{v}_1,\alpha)=\frac{\alpha}{\alpha+1}=\lim_{\bar{v}_1\uparrow 1}\bar{b}(\bar{v}_1,\alpha)$ and $\lim_{\bar{v}_1\uparrow 1}\bar{b}^{*\prime}(\bar{v}_1,\alpha)=\frac{\alpha}{2(\alpha+1)}=\lim_{\bar{v}_1\uparrow 1}\bar{b}'(\bar{v}_1,\alpha)$. Moreover, $\bar{b}^*$ is a convex function of $\bar{v}_1$. If we can prove that $\bar{b}$ is a concave function of $\bar{v}_1$, we will have proved the claim, since both functions are equal and have equal derivatives at $\bar{v}_1=1$. Now,

$$\bar{b}''(\bar{v}_1,\alpha)=\frac{(1+\alpha)\bar{v}_1^{\alpha-1}}{(1-\bar{v}_1^{\alpha+1})^3}\big[(2+\alpha)\bar{v}_1-\alpha+\alpha\bar{v}_1^{\alpha+2}-(2+\alpha)\bar{v}_1^{\alpha+1}\big].$$

Denote by $\kappa(\bar{v}_1,\alpha)$ the term in the last parenthesis of this expression. We want to show that $\kappa<0$ for all $\alpha>0$ and for all $\bar{v}_1\in(0,1)$. Note that $\kappa(0,\alpha)<0$ and $\kappa(1,\alpha)=0$. Thus $\kappa(\bar{v}_1,\alpha)<0$ for all $\bar{v}_1\in(0,1)$ if

$$\frac{\mathrm{d}}{\mathrm{d}\bar{v}_1}\kappa(\bar{v}_1,\alpha)=(\alpha+2)\big(1-(\alpha+1)\bar{v}_1^{\alpha}+\alpha\bar{v}_1^{\alpha+1}\big)>0.$$

This inequality holds because $\frac{\mathrm{d}^2}{\mathrm{d}\bar{v}_1^2}\kappa(\bar{v}_1,\alpha)<0$ and $\frac{\mathrm{d}}{\mathrm{d}\bar{v}_1}\kappa(1,\alpha)=0$. Thus $\bar{b}''(\bar{v}_1,\alpha)<0$ for all $\alpha>0$ and for all $\bar{v}_1\in(0,1)$. Claim 1 follows. □

Claim 2. *$G^*(b)\geqslant G(b)$ for $b\in(\underline{b},\min\{\bar{b},\bar{b}^*\})\Rightarrow\frac{G^{*\prime}(b)}{G^*(b)}<\frac{G'(b)}{G(b)}$.*

Proof of Claim 2. The argument is basically the same as the argument in step 2 of Theorem 4. By definition, $G^*(b)\geqslant G(b)$ implies $\phi^2\geqslant\phi_1\phi_2$.[18] From bidders' FOCs $\frac{G'(b)}{G(b)}=\frac{1}{\phi_1-b}+\frac{1}{\phi_2-b}$

[18] Even if $\phi>\bar{v}_1$ since $F(\phi)^2=F_1(\phi)F_2(\phi)\leqslant\frac{\phi^{2\alpha}}{\bar{v}_1{}^{\alpha}}$.

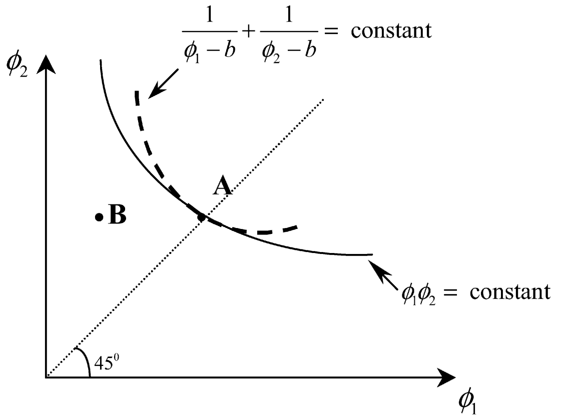

Fig. 2. Iso-level curves.

and $\frac{G^{*\prime}(b)}{G^*(b)} = \frac{2}{\phi(b)-b}$. Figure 2 represents the iso-level curves for $\phi_1\phi_2$ and $\frac{1}{\phi_1-b} + \frac{1}{\phi_2-b}$ in the (ϕ_1, ϕ_2) space. Note that, at $\phi_1 = \phi_2$, the lower contour set of $\frac{1}{\phi_1-b} + \frac{1}{\phi_2-b}$ is contained in the upper contour set of $\phi_1\phi_2$. Consider any b such that $G^*(b) \geqslant G(b)$. Since $\phi_1 \neq \phi_2$, the corresponding values of the inverse bidding functions are represented by A (for $\phi(b), \phi(b)$) and (for example) B (for $\phi_1(b), \phi_2(b)$). Thus $\frac{G'(b)}{G(b)} > \frac{G^{*\prime}(b)}{G^*(b)}$. □

Claims 1 and 2 allow us to complete the argument. By Claim 1, $G^*(b) < G(b)$ close to $\bar{b}$. Claim 2 implies that G^* and G cross at most once. Towards a contradiction, suppose this is the case, that is, $G^*(b) > G(b)$ on the interval $(\underline{b}, \hat{b})$ for some $\hat{b} < \bar{b}$. This implies $\frac{G^{*\prime}(b)}{G^*(b)} \geqslant \frac{G'(b)}{G(b)}$ must hold for some b in $(\underline{b}, \hat{b})$ (indeed, if $\frac{G^{*\prime}(b)}{G^*(b)} < \frac{G'(b)}{G(b)}$ for all $b \in (\underline{b}, \hat{b})$, $G^*(b) < G(b)$ for all $b \in (\underline{b}, \hat{b})$). Claim 2 then implies $G^*(b) < G(b)$ at this point, a contradiction. Thus $G^*(b) < G(b)$ for all b. □

Appendix B. Computations for Example 4

We need to compute

$$R = \int_4^8 \int_4^8 \max\{J(v_1), J(v_2), 0\} f(v_1) f(v_2)\, \mathrm{d}v_1\, \mathrm{d}v_2,$$

where

$$f(v) = \begin{cases} \frac{1}{2}, & v \in [4, 5], \\ \frac{1}{4\sqrt{v-4}}, & v > 5 \end{cases}$$

and

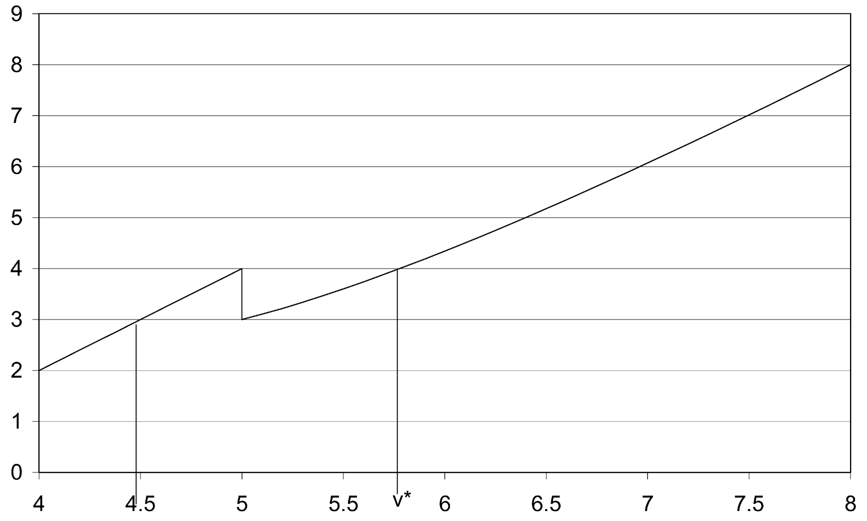

Fig. 3. Virtual valuation in the symmetric benchmark.

$$J(v) = \begin{cases} 2v - 6 & \text{for } v \in [4, 5], \\ 3v - 8 - 4\sqrt{v-4} & \text{for } v > 5 \end{cases}$$

(J is represented in Fig. 3). Since $J(v) > 0$ and since the environment is symmetric, R can be rewritten as

$$R = 2\int_4^8 J(v)\Pr\bigl(J(\hat{v}) < J(v)\bigr) f(v)\,\mathrm{d}v, \tag{B.23}$$

where $\Pr(J(\hat{v}) < J(v))$ stands for the probability that a valuation draw from F has a virtual valuation lower than $J(v)$.

Define v^* such $J(v^*) = 3v - 8 - 4\sqrt{v-4} = 4$ (see Fig. 3). We have $v^* = \frac{52}{9}$ and $J(\cdot)$ monotonically increasing over $[\frac{52}{9}, 8]$. Therefore, $\Pr(J(\hat{v}) < J(v)) = F(v)$ for all $v \geqslant v^*$. Similarly, define v^{**} such that $2v^{**} - 6 = 3$. We have $v^{**} = 4.5$ and $\Pr(J(\hat{v}) < J(v)) = F(v)$ for $v \leqslant 4.5$.

To compute the expression for $\Pr(J(\hat{v}) < J(v))$ when $v \in (v^{**}, v^*)$, we proceed as follows. When $v \in (v^{**}, 5)$, $\Pr(J(\hat{v}) < J(v)) = F(v) + F(v_2) - F(5)$, where $v_2 > 5$ is such that $J(v_2) = J(v)$. When $v \in (5, v^{**})$, $\Pr(J(\hat{v}) < J(v)) = F(v_1) + F(v) - F(5)$, where $v_1 < 5$ is such that $J(v_1) = J(v)$.

$$v \in \bigl(v^{**}, 5\bigr): \quad J(v_2) = 3v_2 - 8 - 4\sqrt{v_2 - 4} = 2v - 6$$
$$\Leftrightarrow \quad v_2 = \frac{14}{9} + \frac{4}{9}\sqrt{6v - 26} + \frac{2}{3}v,$$
$$v \in \bigl(5, v^{**}\bigr): \quad J(v_1) = 2v_1 - 6 = 3v - 8 - 4\sqrt{v-4} \quad \Leftrightarrow \quad v_1 = \frac{3}{2}v - 1 - 2\sqrt{v-4}.$$

We can now rewrite (B.23) as

$$R = 2\int_{4}^{4.5} \frac{(v-3)(v-4)}{2}\,\mathrm{d}v + 2\int_{4.5}^{\frac{52}{9}} f(v)J(v)\Pr\big(J(\hat{v}) < J(v)\big)\,\mathrm{d}v$$

$$+ \frac{1}{4}\int_{\frac{52}{9}}^{8} \big(3v - 8 - 4\sqrt{v-4}\big)\,\mathrm{d}v$$

$$= 3.4507 + \int_{4.5}^{5} (2v-6)\left(\frac{v-4}{2} + \frac{1}{2}\sqrt{-\frac{22}{9} + \frac{4}{9}\sqrt{6v-26} + \frac{2}{3}v - \frac{1}{2}}\right)\mathrm{d}v$$

$$+ \int_{5}^{\frac{52}{9}} \big(3v - 8 - 4\sqrt{v-4}\big)\frac{1}{2\sqrt{v-4}}\left(\frac{3}{4}v - 3 - \frac{1}{2}\sqrt{v-4}\right)\mathrm{d}v$$

$$= 3.4507 + 0.83549 + 0.52191$$

$$= 4.8081.$$

(The integrals were evaluated numerically with Maple in Scientific Workplace.)

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